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A Complex Fuzzy MAGDM Framework for Sustainable Gold Mining using Hamacher Aggregation Operators

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ABSTRACT

Cleaner Production (CP) is widely recognized as a key strategy for balancing environmental protection with economic development in industrial sectors, including gold mining. In this study, we propose a novel multi-attribute group decision-making (MAGDM) model based on Complex p, q-Rung Orthopair Fuzzy Sets (Cp,q-$\mathcal{R}$OFSs) and Hamacher Aggregation Operators (HAOs) to evaluate sustainability-driven decisions in gold mining operations. The model captures the complex and uncertain nature of expert assessments using complex-valued membership structures and flexible aggregation processes. To demonstrate the model's practical utility, a real-world-inspired case study involving the evaluation of five cleaner production alternatives in a gold mining scenario is conducted. These alternatives are assessed based on environmental, economic, and technical criteria. The proposed framework effectively aggregates expert opinions under uncertainty, and a detailed comparative and sensitivity analysis validates the robustness and precision of the method. Results show that the model supports informed and sustainable decision-making in mining practices, offering a promising tool for industries seeking to implement CP under uncertain and complex conditions.

1. Introduction

Real-world decision-making problems often involve multiple conflicting criteria and require inputs from multiple experts. This framework is referred to as Multi-Attribute Group Decision Making (MAGDM) and is widely applied in fields such as engineering, medical diagnostics, financial analysis, and artificial intelligence. MAGDM techniques aim to evaluate and rank various alternatives based on multiple attributes, despite uncertainty, vagueness, or imprecise data. The engineering, financial risk management, medical diagnosis, artificial intelligence, and other related fields frequently struggle with making decisions based on insufficient information. Fuzzy Sets (FSs), which Zadeh (1965) [1] introduced in connection with uncertainty in decision-making situations, use a membership function with values between 0 and 1 to resolve uncertainty. Intuitionistic Fuzzy Sets

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(IFS), developed by Atanassov in the year 1986 [2], set membership and non-membership degrees based on the restriction that their sum cannot exceed one because FSs cannot resolve any kind of hesitancy or partial information. Even though IFSs have improved their method of thinking about FSs [3], there will still be some boundaries of greater vagueness that cannot be resolved.

Yager (2013) [4] suggested Pythagorean FS (PFS) as a solution to this problem. These sets offer a more flexible structure for decision-making [5] by ensuring that the sum of squares of membership and non-membership degrees stays inside [0,1]. Using a q-power sum constraint, Yager (2016) [6] further developed this concept by introducing q-Rung orthopair Fuzzy Sets (q-$\mathcal{R}$OFSs), which provide more flexibility in representing uncertainty and imprecise information. To increase the q-$\mathcal{R}$OFS's applicability in Multi-Attribute Group Decision-Making (MAGDM) issues, two independent power parameters (p and q) were added to create the p, q-Rung orthopair Fuzzy Set (p,q-$\mathcal{R}$OFS) ([7],[8]), which was recently introduced.

Nevertheless, there are some inherent characteristics of real-life systems that cannot be handled by the basic fuzzy models, like periodicity, oscillatory behavior, or phase-based uncertainty. To overcome this limitation, Ramot *et al.* (2002) [9] presented CFSs to extend the FSs by including the phase aspect of uncertainty that includes real and imaginary functions for the membership degrees. After the generalization of PFSs to complex-valued membership functions termed as CPFSs. The new set called CIFSs was introduced by Alkouri *et al.* (2012) [10] to include the intuitionistic vagueness.

To enrich these models, more fuzziness was added into the models while maintaining phase-related ambiguity with the help of the development of complex q-rung orthopair Fuzzy Sets (Cq-$\mathcal{R}$OFSs). The reason for proposing the Cp,q-rung orthopair Fuzzy Sets was because these models are incapable of dealing with orthopair constraints of higher order real and imaginary components simultaneously. This new addition helps in broadening the capability of the model in handling uncertainties in MAGDM situations by adding p, q-power constraints to the real and imaginary parts of the MFs.

In the case of IFSs, measures of similarity have been defined to compare the options and enhance the decision-making (Dengfeng *et al.*, 2002; Ye, 2011) [11]. To enhance information fusion, many aggregation operators have also been developed; for instance, the Einstein Choquet integral operator (Xu *et al.*, 2014) [12]. Moreover, several MCDM approaches have been employed in intuitionistic fuzzy contexts like GRA-based selection models (Zhang *et al.* 2011) [13], TOPSIS based decision models (Boran *et al.*, 2009) [14], and decision frameworks based on ELECTRE (Devi *et al.* 2013) [15]. However, the ability of high-order fuzzy models to deal with phase-related uncertainty is not fully realized by these current methods, which are developed under the real-valued fuzzy context.

Despite these advancements, classical fuzzy models lack the capacity to represent phase-based or periodic uncertainty. To address this, Complex Fuzzy Sets (CFSs) [9] introduced complex-valued membership functions. This led to subsequent developments such as Complex Pythagorean Fuzzy Sets (CPFSs) and Complex Intuitionistic Fuzzy Sets (CIFSs) [10]. The latest evolution Complex p,q-Rung Orthopair Fuzzy Sets (Cp,q-ROFSs) adds real and imaginary components governed by p- and q-power constraints, making them particularly suitable for MAGDM problems with rich and uncertain information.

To enhance the usability of the existing fuzzy models in MAGDM contexts, the overall purpose of this research is to develop Cp,q-$\mathcal{R}$OFSs, which extend complicated fuzzy forms. Aggregate operators are also required to consolidate multiple criteria, but non-linear interactions are often neglected by arithmetic and geometric mean methods. Therefore, this paper introduces HAOs for Cp,q-$\mathcal{R}$OFSs to increase the flexibility in dealing with complex decision-making problems based on Hamacher t-norm and t-conorm functions [16]. They are used in decision-making, especially in scenarios that are ambiguous. The presented approach is especially useful when one must solve MAGDM problems, which occur when decision-makers are to choose between multiple options based on multiple criteria. Figure 1 below summarizes the evolution from classical FSs to the advanced Cp,q-ROFSs framework:

Fuzzy Model	**Key Feature**	**Introduced By**
Fuzzy Sets (FSs)	Basic membership value [0,1]	Zadeh (1965)
Intuitionistic FSs	Membership + Non-membership + Hesitancy	Atanassov (1986)
Pythagorean FSs	Sum of squares ≤ 1	Yager (2013)
q-Rung Orthopair FSs	Generalized sum of powers ≤ 1	Yager (2016)
p,q-ROFSs	Dual-power flexibility (p and q constraints)	Recent advancement
Complex FSs	Real + imaginary membership components	Ramot et al. (2002)
Complex Intuitionistic FSs	Complex-valued membership and non-membership	Alkouri et al. (2012)
Cp,q-ROFSs	Complex-valued + dual power constraints	This study

Figure 1: Cp,q-ROFSs framework:

In recent years, the development of advanced decision-making models under uncertainty has gained momentum, particularly through the integration of fuzzy, neutrosophic, and probabilistic hesitant frameworks. For instance, an optimization strategy under a probabilistic neutrosophic hesitant fuzzy rough environment has been proposed to enhance confidence-level-based MADM decisions, enabling robust handling of vagueness and reliability [21]. Similarly, the neutrosophic Z-rough set approach combined with sine trigonometric aggregation operators has proven effective for evaluating sustainable industrial alternatives [22], addressing multidimensional uncertainty through refined set-theoretic operations.

Moreover, recent studies have introduced single-valued neutrosophic probabilistic hesitant fuzzy rough aggregation models for complex real-world applications such as smart city planning, further illustrating the importance of hybrid decision models in uncertain and dynamic environments [23]. These innovative models demonstrate the utility of combining multiple fuzzy logic extensions to better reflect human reasoning under incomplete, inconsistent, and hesitant information.

Inspired by these advancements, our study proposes a Complex p, q -Rung Orthopair Fuzzy Hamacher Aggregation Model for MAGDM, tailored for sustainable gold mining evaluation. This framework distinguishes itself by capturing both the interactive behavior of criteria through Hamacher operations and the complex nature of expert judgments, extending beyond traditional intuitionistic and neutrosophic systems.

The introduction of Cp,q-$\mathcal{R}$OFSs, which extend complex-valued fuzzy frameworks by adding p,q-power constraints on both real and imaginary components, is one of the study's main achievements. It also creates Hamacher Aggregation Operators (HAOs) for Cp,q-$\mathcal{R}$OFSs, which allow for more sophisticated aggregation methods and better decision-making. Additionally, this paper applies the suggested method to a real-world MAGDM problem assessing the effectiveness of exploration and recovery robots and performs comparison analysis to show the superiority of Cp,q-$\mathcal{R}$OFS-based aggregation methods over existing models. In light of this development, the present study proposes novel Hamacher Aggregation Operators (HAOs) for Cp,q-ROFSs to effectively aggregate information in complex MAGDM scenarios. HAOs based on t-norms and t-conorms offer more realistic aggregation behavior than conventional arithmetic means, especially in uncertain environments.

Recent years have seen a surge in advanced fuzzy decision-making techniques aimed at managing uncertainty in group decision contexts. One notable stream of work has focused on the Best-Worst Method (BWM)**,** especially in fuzzy environments. For instance, Rashid et al. (2023) conducted a comprehensive review of fuzzy BWM models with a focus on human-centric decision-making, emphasizing their applicability in group evaluation frameworks [24]. Further, Pythagorean Fuzzy Sets (PFSs) have emerged as a powerful extension of Intuitionistic Fuzzy Sets (IFSs), enabling greater flexibility in modeling uncertain information. A detailed survey by Garg et al. (2021) outlines the theoretical advancements and practical applications of PFSs from 2013 to 2020 [25], indicating their growing relevance in sustainability and risk-based assessments. Similarly, decision-making models that extend the Analytic Hierarchy Process (AHP) into fuzzy environments have received extensive attention. Stanujkic et al. (2023) reviewed fuzzy extensions of AHP, providing a clear overview of how fuzzy logic enhances traditional hierarchical structuring in MADM problems [26]. In addition to survey-based research, practical applications of hybrid fuzzy methods continue to expand. For example, Karabasevic et al. (2019) proposed a hybrid fuzzy AHP–TOPSIS model for warehouse location selection, showcasing how multi-operator frameworks can be tailored for real-world logistics decisions [27]. Moreover, the evolution of generalized fuzzy numbers is another notable area. Mardani et al. (2020) conducted a systematic review on generalized fuzzy numbers, mapping their theoretical development and widespread applications across disciplines [28]. These insights align well with our aim to extend decision-making theory using complex-valued fuzzy structures like Cp,q-$\mathcal{R}$OFSs**.** In this context, our proposed framework, grounded in complex fuzzy theory and Hamacher aggregation—offers a novel and flexible alternative to existing fuzzy MADM models. It extends the current literature by integrating complex membership representations and advanced fusion techniques for sustainability-focused decisions.

The structure of the paper is as follows: HAOs, p,q-Rung Orthopair Fuzzy Sets (p,q-$\mathcal{R}$OFSs), and their characteristics are covered in Section 2. In Section 3, Cp,q-$\mathcal{R}$OFSs are introduced with attributes, including averaging and geometric HAOs, and are examined. Section 4 investigates the effects of changing Hamacher aggregation parameters and applies the suggested method to MAGDM. A comparison investigation demonstrates the new operators' advantages. Section 5 concludes with a summary of the main conclusions and recommendations for further study. This paper also applies the proposed HAOs to a real-world MAGDM case involving the evaluation of exploration and recovery robots. The results demonstrate the superiority of the Cp,q-ROFS-based approach over existing fuzzy models.

2. Methodology

The concept of C*p,q*-$\mathcal{R}$OFSs and its basic characteristics will be covered in this section.

Definition 1: [7] On X a *p,q*-$\mathcal{R}$OFSs is described as:

$$\mathrm{E} = \{(x, \acute{m}_E(x), \acute{n}_E(x)) : x \in \mathrm{X}\} \tag{1}$$

which satisfies the subsequent requirement: $0 \leq \acute{m}_E^p(x) + \acute{n}_E^q(x) \leq 1$. Where $\acute{m}_E(x), \acute{n}_E(x) \in [0,1]$. The truth degree is represented by the symbol $\acute{m}_E(x)$ whereas the falsity degree is represented by the symbol $\acute{n}_E(x)$. The notation for the *p,q*-$\mathcal{R}$OFN is $\acute{m}_E(x), \acute{n}_E(x), p \neq q$. Where,

i. $\acute{m}_E(x) \in [0,1]$ represents the **truth-membership degree** of element x
ii. $\acute{n}_E(x) \in [0,1]$ represents the **falsity-membership degree** of element x
iii. $p, q > 0$ are the **rung parameters** controlling flexibility.

Definition 2: For *p,q*-$\mathcal{R}$OFN $Ą = \acute{m}_E(x), \acute{n}_E(x)$, the score function is provided by

$$\acute{\check{S}}(Ą) = \acute{m}_E^p(x)\acute{n}_E^q(x), \acute{\check{S}}(Ą) \in [-1, 1] \tag{2}$$

Definition 3: For *p,q*-$\mathcal{R}$OFN $Ą = \acute{m}_E(x), \acute{n}_E(x)$, the accuracy function is provided by

$$H(Ą) = \acute{m}_E^p(x) + \acute{n}_E^q(x), H(Ą) \in [0, 1] \tag{3}$$

Definition 4: Consider the two *p,q*-$\mathcal{R}$OFNs. $Ą_1 = (\acute{m}_1(x), \acute{n}_1(x))$ and $Ą_2 = (\acute{m}_2(x), \acute{n}_2(x))$ then by using the Def. (2 & 3), we have two functions:

i. Score function: $\acute{\check{S}}(Ą_1) = \acute{m}_1^p(x) + \acute{n}_1^q(x)\ and\ \acute{\check{S}}(Ą_2) = \acute{m}_2^p(x) + \acute{n}_2^q(x)$,
ii. Accuracy function: $H(Ą_1) = (\acute{m}_1^p(x) + \acute{n}_1^q(x))\ and\ H(Ą_2) = (\acute{m}_2^p(x) + \acute{n}_2^q(x))$

If score function is defined as: $\acute{\check{S}}(Ą_2) < \acute{\check{S}}(Ą_1)$ then we have $Ą_2 < Ą_1$

If score function of $Ą_1\ and\ Ą_2$ is defined as: $\acute{\check{S}}(Ą_2) = \acute{\check{S}}(Ą_1)$ .Then we will move to the accuracy function and

a) If $H(Ą_2) < H(Ą_1)$ then $Ą_2$ will be greater than $Ą_1$.
b) If $H(Ą_2) = H(Ą_1)$ then $Ą_2$ will be equal to $Ą_1$.

Definition 5: Consider a finite universal set X, a C*p,q*-$\mathcal{R}$OFS E is defined as:

$$E = \{(X, \acute{m}_E(x), \acute{n}_E(x)) : x \in X\} \tag{4}$$

which satisfies the subsequent requirement: $0 \leq m_E^p(\tilde{v}) + n_E^q(\tilde{v}) \leq 1$ and $0 \leq \varphi_{m_E}^p(\tilde{v}) + \varphi_{n_E}^q(\tilde{v}) \leq 1$, where $m_E(\tilde{v}) + n_E(\tilde{v}) \in [0,1]$. The symbols $\acute{m}_E(\tilde{v}) = m_E(\tilde{v}).e^{\acute{i}\,2\pi\varphi_{m_E(\tilde{v})}}$, $\acute{n}_E(\tilde{v}) = n_E(\tilde{v}).e^{\acute{i}\,2\pi\varphi_{n_E(\tilde{v})}}$ are symbolized by the complexed-valued truth degree and falsity degree respectively. The C*p,q*-$\mathcal{R}$OFN is given by $Ą = (\acute{m}_E, \acute{n}_E) = \left(m_E(\tilde{v}).e^{\acute{i}\,2\pi\varphi_{m_E(\tilde{v})}}, n_E(\acute{\tilde{v}}).e^{\acute{i}\,2\pi\varphi_{n_E(\tilde{v})}}\right)$. Where,

i. $\acute{m}_E(x) \in C$ is the complex-valued truth-membership degree
ii. $\acute{n}_E(x) \in C$ is the complex-valued falsity-membership degree

Definition 6: The score function of $Ą = \left(m_E(\tilde{v}).e^{\acute{i}\,2\pi\varphi_{m_E(\tilde{v})}}, n_E(\tilde{v}).e^{\acute{i}\,2\pi\varphi_{n_E(\tilde{v})}}\right)$ is defined as:

$$\acute{\acute{S}}(Ą) = \frac{1}{4}\left(2 + m_1^p - n_1^q + \varphi_{m_1}^p - \varphi_{n_1}^q\right) \tag{5}$$

Definition 7: The accuracy function of $Ą = \left(m_E(\tilde{v}).e^{\acute{i}\,2\pi\varphi_{m_E(\tilde{v})}}, n_E(\tilde{v}).e^{\acute{i}\,2\pi\varphi_{n_E(\tilde{v})}}\right)$ is defined as:

$$H(Ą) = \frac{1}{2}\left(m_E^p + n_E^q + \varphi_{m_E}^p + \varphi_{n_E}^q\right) \tag{6}$$

Definition 8: let's have two C*p,q*-$\mathcal{R}$OFNs

$$Ą_1 = \left(m_1(\tilde{v}).e^{\acute{i}\,2\pi\varphi_{m_1(\tilde{v})}}, n_1(\tilde{v}).e^{\acute{i}\,2\pi\varphi_{n_1(\tilde{v})}}\right)$$

And

$$Ą_2 = \left(m_2(\tilde{v}).e^{\acute{i}\,2\pi\varphi_{m_2(\tilde{v})}}, n_2(\tilde{v}).e^{\acute{i}\,2\pi\varphi_{n_2(\tilde{v})}}\right)$$

then by Eq. (5) and Eq. (6). We have score and accuracy function of $Ą_1$ and $Ą_2$

$$\acute{\acute{S}}(Ą_1) = \frac{1}{4}\left(2 + m_1^p - n_1^q + \varphi_{m_1}^p - \varphi_{n_1}^q\right),$$
$$\acute{\acute{S}}(Ą_2) = \frac{1}{4}\left(2 + m_2^p - n_2^q + \varphi_{m_2}^p - \varphi_{n_2}^q\right),$$
$$H(Ą_1) = \frac{1}{2}\left(m_1^p + n_1^q + \varphi_{m_1}^p + \varphi_{n_1}^q\right)$$
$$H(Ą_2) = \frac{1}{2}\left(m_2^p + n_2^q + \varphi_{m_2}^p + \varphi_{n_2}^q\right)$$

i. If the score function is: $\acute{\acute{S}}(Ą_2) < \acute{\acute{S}}(Ą_1)$ then we have $Ą_2 < Ą_1$.
ii. If the score function of $Ą_1$ *and* $Ą_2$ is: $\acute{\acute{S}}(Ą_2) = \acute{\acute{S}}(Ą_1)$ then we will move to accuracy function.
   a) If $H(Ą_2) < H(Ą_1)$ then $Ą_2$ will be greater than $Ą_1$.
   b) If $H(Ą_2) = H(Ą_1)$ then $Ą_2 = Ą_1$.

Here, we have two C*p,q*-ROFNs $Ą_1 = \left(0.7e^{\iota 2\pi(0.76)}, 0.69e^{\iota 2\pi(0.76)}\right)$ and $Ą_2 = \left(0.75e^{\iota 2\pi(0.80)}, 0.74e^{\iota 2\pi(0.79)}\right)$,from Eq. (5) and *p* =3.0 *q* = 4.0

$\acute{\acute{S}}(Ą_1) = \frac{1}{4}(2 + 0.7^3 - 0.69^4 + 0.77^3 - 0.76^4) = \frac{1}{4}(2 + .24) = 0.5$

$\acute{\check{S}}(\text{Ą}_2) = \frac{1}{4}(2 + 0.75^3 - 0.74^4 + 0.80^3 - 0.79^4) = \frac{1}{4}(2 + .30) = 0.5$

Consequently, $S(\text{Ą}_2) = S(\text{Ą}_1)$. Next, we'll apply Eq. (6) so that

$$H(\text{Ą}_1) = \frac{1}{2}(0.7^3 - 0.69^4 + 0.77^3 - 0.76^4) = \frac{1}{2}(0.239) = 0.119$$

$$H(\text{Ą}_2) = \frac{1}{2}(0.75^3 - 0.74^4 + 0.80^3 - 0.79^4) = \frac{1}{2}(0.244) = 0.122$$

Consequently, $H(\text{Ą}_2) > H(\text{Ą}_1)\ then\ \text{Ą}_2 > \text{Ą}_1$.

3. Results

3.1 Hamacher operators for Complex p,q-rung orthopair fuzzy sets

Hamacher operations [17] superiority for CIFNs, CPFNs and CFNs over those in C*p,q*-$\mathcal{R}$OF environments is highlighted in this section. The Hamacher *t*-norm and *t*-conorm are used to introduce HAOs [16].

Definition 9: Consider a pair $\text{Ą} = \left(m_{\text{Ą}}\, e^{i2\pi\varpi_{m_{\text{Ą}}}}, n_{\text{Ą}} e^{i2\pi\varpi_{n_{\text{Ą}}}}\right)$ and $B = \left(m_B\, e^{i2\pi\varpi_{m_B}}, n_B e^{i2\pi\varpi_{n_B}}\right)$

and for $\lambda > 0$. The fuzzy Hamacher operations for C*p,q*-$\mathcal{R}$Os are:

1. $$\text{Ą}\oplus B = \left(\begin{array}{c} \sqrt[p]{\frac{m_{\text{Ą}}^p + m_B^p - m_{\text{Ą}}^p m_B^p - (1-\gamma) m_{\text{Ą}}^p m_B^p}{1-(1-\gamma) m_{\text{Ą}}^p m_B^p}}\; e^{i2\pi\left(\sqrt[p]{\frac{\varpi_{m_{\text{Ą}}}^p + \varpi_{m_B}^p - \varpi_{m_{\text{Ą}}}^p \varpi_{m_B}^p - (1-\gamma)\varpi_{m_{\text{Ą}}}^p \varpi_{m_B}^p}{1-(1-\gamma)\varpi_{m_{\text{Ą}}}^p \varpi_{m_B}^p}}\right)} \\ \frac{n_{\text{Ą}} n_B}{\sqrt[q]{\gamma + (1-\gamma)\left(n_{\text{Ą}}^q + n_B^q - n_{\text{Ą}}^q n_B^q\right)}}\; e^{i2\pi\left(\frac{\varpi_{n_{\text{Ą}}}^q \varpi_{n_B}^q}{\sqrt[q]{\gamma+(1-\gamma)\left(\varpi_{n_{\text{Ą}}}^q + \varpi_{n_B}^q - \varpi_{n_{\text{Ą}}}^q \varpi_{n_B}^q\right)}}\right)} \end{array} \right), \gamma > 0$$

2. $$\text{Ą}\otimes B = \left(\begin{array}{c} \frac{m_{\text{Ą}} m_B}{\sqrt[p]{\gamma + (1-\gamma)\left(m_{\text{Ą}}^p + m_B^p - m_{\text{Ą}}^p m_B^p\right)}}\; e^{i2\pi\left(\frac{\varpi_{m_{\text{Ą}}}^p \varpi_{m_B}^p}{\sqrt[p]{\gamma+(1-\gamma)\left(\varpi_{m_{\text{Ą}}}^p + \varpi_{m_B}^p - \varpi_{m_{\text{Ą}}}^p \varpi_{m_B}^p\right)}}\right)} \\ \sqrt[q]{\frac{n_{\text{Ą}}^q + n_B^q - n_{\text{Ą}}^q n_B^q - (1-\gamma) n_{\text{Ą}}^q n_B^q}{1-(1-\gamma) n_{\text{Ą}}^q n_B^q}}\; e^{i2\pi\left(\sqrt[q]{\frac{\varpi_{n_{\text{Ą}}}^q + \varpi_{n_B}^q - \varpi_{n_{\text{Ą}}}^q \varpi_{n_B}^q - (1-\gamma)\varpi_{n_{\text{Ą}}}^q \varpi_{n_B}^q}{1-(1-\gamma)\varpi_{n_{\text{Ą}}}^q \varpi_{n_B}^q}}\right)} \end{array} \right), \gamma > 0$$

3. $$\lambda \text{Ą} = \left(\begin{array}{c} \sqrt[p]{\frac{\left(1+(\gamma-1)m_{\text{Ą}}^{p}\right)^{\lambda}-\left(1-m_{\text{Ą}}^{p}\right)^{\lambda}}{\left(1+(\gamma-1)m_{\text{Ą}}^{p}\right)^{\lambda}+(\gamma-1)\left(1-m_{\text{Ą}}^{p}\right)^{\lambda}}}\, e^{i2\pi\left(\sqrt[p]{\frac{\left(1+(\gamma-1)\varpi_{m_{\text{Ą}}^{p}}\right)^{\lambda}-\left(1-\varpi_{m_{\text{Ą}}^{p}}\right)^{\lambda}}{\left(1+(\gamma-1)\varpi_{m_{\text{Ą}}^{p}}\right)^{\lambda}+(\gamma-1)\left(1-\varpi_{m_{\text{Ą}}^{p}}\right)^{\lambda}}}\right)} \\ \frac{\sqrt[q]{\gamma}\, n_{\text{Ą}}^{\lambda}}{\sqrt[q]{\left(1+(\gamma-1)\left(1-n_{\text{Ą}}^{q}\right)\right)^{\lambda}+(\gamma-1)\left(n_{\text{Ą}}^{q}\right)^{2\lambda}}}\, e^{i2\pi\left(\frac{\sqrt[q]{\gamma}\,\varpi_{m_{\text{Ą}}}^{\lambda}}{\sqrt[q]{\left(1+(\gamma-1)\left(1-\varpi_{n_{\text{Ą}}^{q}}\right)\right)^{\lambda}+(\gamma-1)\varpi_{n_{\text{Ą}}}^{q\;2\lambda}}}\right)} \end{array} \right), \gamma > 0$$

4. $$\text{Ą}^{\lambda} = \left(\begin{array}{c} \frac{\sqrt[p]{\gamma}\, m_{\text{Ą}}^{\lambda}}{\sqrt[p]{\left(1+(\gamma-1)\left(1-m_{\text{Ą}}^{p}\right)\right)^{\lambda}+(\gamma-1)\left(m_{\text{Ą}}^{p}\right)^{2\lambda}}}\, e^{i2\pi\left(\frac{\sqrt[p]{\gamma}\,\varpi_{m_{\text{Ą}}}^{\lambda}}{\sqrt[p]{\left(1+(\gamma-1)\left(1-\varpi_{m_{\text{Ą}}^{p}}\right)\right)^{\lambda}+(\gamma-1)\varpi_{m_{\text{Ą}}}^{p\;2\lambda}}}\right)} \\ \frac{\sqrt[q]{\gamma n_{\text{Ą}}^{\lambda}}}{\sqrt[q]{\left(1+(\gamma-1)\left(1-n_{\text{Ą}}^{q}\right)\right)^{\lambda}+(\gamma-1)\left(n_{\text{Ą}}^{q}\right)^{2\lambda}}}\, e^{i2\pi\left(\sqrt[q]{\frac{\left(1+(\gamma-1)\varpi_{n_{\text{Ą}}^{q}}\right)^{\lambda}-\left(1-\varpi_{n_{\text{Ą}}^{q}}\right)^{\lambda}}{\left(1+(\gamma-1)\varpi_{n_{\text{Ą}}^{q}}\right)^{\lambda}+(\gamma-1)\left(1-\varpi_{n_{\text{Ą}}^{q}}\right)^{\lambda}}}\right)} \end{array} \right), \gamma > 0$$

To illustrate Def 9, using an example, we have pair of $\text{Ą} = \left(0.7e^{i2\pi(0.77)}, 0.69e^{i2\pi(0.76)}\right)$ and $B = \left(0.75e^{i2\pi(0.80)}, 0.74e^{i2\pi(0.79)}\right)$. Then by using Def. (9) where $p = 3,\ q = 4\ and\ \lambda = 2$.

1. $$\text{Ą} \oplus B = \left(\begin{array}{c} \sqrt[p]{\frac{m_{\text{Ą}}^{p}+m_{B}^{p}-m_{\text{Ą}}^{p}m_{B}^{p}-(1-\gamma)m_{\text{Ą}}^{p}m_{B}^{p}}{1-(1-\gamma)m_{\text{Ą}}^{p}m_{B}^{p}}}\, e^{i2\pi\left(\sqrt[p]{\frac{\varpi_{m_{\text{Ą}}^{p}}+\varpi_{m_{B}^{p}}-\varpi_{m_{\text{Ą}}^{p}}\varpi_{m_{B}^{p}}-(1-\gamma)\varpi_{m_{\text{Ą}}^{p}}\varpi_{m_{B}^{p}}}{1-(1-\gamma)\varpi_{m_{\text{Ą}}^{p}}\varpi_{m_{B}^{p}}}}\right)} \\ \frac{n_{\text{Ą}}n_{B}}{\sqrt[q]{\gamma+(1-\gamma)\left(n_{\text{Ą}}^{q}+n_{B}^{q}-n_{\text{Ą}}^{q}n_{B}^{q}\right)}}\, e^{i2\pi\left(\frac{\varpi_{n_{\text{Ą}}^{q}}\varpi_{n_{B}^{q}}}{\sqrt[q]{\gamma+(1-\gamma)\left(\varpi_{n_{\text{Ą}}^{q}}+\varpi_{n_{B}^{q}}-\varpi_{n_{\text{Ą}}^{q}}\varpi_{n_{B}^{q}}\right)}}\right)} \end{array} \right)$$

$$= \begin{pmatrix} \sqrt[3]{\frac{0.7^3 + 0.75^3 - 0.7^3 \times 0.75^3 - (1-2) \times 0.7^3 \times 0.75^3}{1-(1-2) \times 0.7^3 \times 0.75^3}}\, e^{i2\pi\left(\sqrt[3]{\frac{0.77^3+0.80^3-0.77^3\times 0.80^3-(1-2)\times 0.77^3\times 0.80^3}{1-(1-2)\times 0.77^3\times 0.80^3}}\right)} \\ \frac{0.69 \times 0.74}{\sqrt[4]{2+(1-2)(0.69^4+0.74^4-0.69^4\times 0.74^4)}}\, e^{i2\pi\left(\frac{0.76\times 0.79}{\sqrt[4]{2+(1-2)(0.76^4+0.79^4-0.76^4\times 0.79^4)}}\right)} \end{pmatrix}$$

$$= \begin{pmatrix} \sqrt[3]{\frac{0.764}{1.144}}\, e^{i2\pi\left(\sqrt[3]{\frac{0.968}{1.244}}\right)} \\ \frac{0.51}{\sqrt[4]{1.54}}\, e^{i2\pi\left(\frac{0.6}{\sqrt[4]{2.56}}\right)} \end{pmatrix}$$

$$= \begin{pmatrix} 0.874 e^{i2\pi(0.92)} \\ 0.13 e^{i2\pi(0.47)} \end{pmatrix}$$

$$2.\ Ą \otimes B = \begin{pmatrix} \frac{m_Ą m_B}{\sqrt[p]{\gamma + (1-\gamma)\left(m_Ą^p + m_B^p - m_Ą^p m_B^p\right)}}\, e^{i2\pi\left(\frac{\varpi_{m_Ą}^p \varpi_{m_B}^p}{\sqrt[p]{\gamma+(1-\gamma)\left(\varpi_{m_Ą}^p+\varpi_{m_B}^p-\varpi_{m_Ą}^p\varpi_{m_B}^p\right)}}\right)} \\ \sqrt[q]{\frac{n_Ą^q + n_B^q - n_Ą^q n_B^q - (1-\gamma) n_Ą^q n_B^q}{1-(1-\gamma) n_Ą^q n_B^q}}\, e^{i2\pi\left(\sqrt[q]{\frac{\varpi_{n_Ą}^q+\varpi_{n_B}^q-\varpi_{n_Ą}^q\varpi_{n_B}^q-(1-\gamma)\varpi_{n_Ą}^q\varpi_{n_B}^q}{1-(1-\gamma)\varpi_{n_Ą}^q\varpi_{n_B}^q}}\right)} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{0.7 \times 0.75}{\sqrt[3]{2+(1-2)(0.7^3+0.75^3-0.7^3\times 0.75^3)}}\, e^{i2\pi\left(\frac{0.77\times 0.80}{\sqrt[3]{2+(1-2)(0.77^3+0.80^3-0.77^3\times 0.80^3)}}\right)} \\ \sqrt[4]{\frac{(0.69^4+0.74^4-0.69^4\times 0.74^4)-(1-2)\times 0.69^4\times 0.74^4}{1-(1-2)\times 0.69^4\times 0.74^4}}\, e^{i2\pi\left(\sqrt[4]{\frac{0.76^4+0.79^4-0.76^4\times 0.79^4-(1-2)\times 0.76^4\times 0.79^4}{1-(1-2)\times 0.76^4\times 0.79^4}}\right)} \end{pmatrix}$$

$$= \left(\frac{0.53}{1.113} e^{i2\pi\left(\frac{0.616}{1.081}\right)}, \sqrt[4]{\frac{0.53}{1.06}}\, e^{i2\pi\left(\sqrt[4]{\frac{0.72}{1.13}}\right)}\right)$$

$$= \begin{pmatrix} 0.47 e^{i2\pi(0.569)} \\ 0.84 e^{i2\pi(0.89)} \end{pmatrix}$$

$$3.\ \lambda \text{Ą} = \begin{pmatrix} \sqrt[p]{\frac{\left(1+(\gamma-1)m_{\text{Ą}}^{p}\right)^{\lambda}-\left(1-m_{\text{Ą}}^{p}\right)^{\lambda}}{\left(1+(\gamma-1)m_{\text{Ą}}^{p}\right)^{\lambda}+(\gamma-1)\left(1-m_{\text{Ą}}^{p}\right)^{\lambda}}}\, e^{i2\pi\left(\sqrt[p]{\frac{\left(1+(\gamma-1)\varpi_{m_{\text{Ą}}^{p}}\right)^{\lambda}-\left(1-\varpi_{m_{\text{Ą}}^{p}}\right)^{\lambda}}{\left(1+(\gamma-1)\varpi_{m_{\text{Ą}}^{p}}\right)^{\lambda}+(\gamma-1)\left(1-\varpi_{m_{\text{Ą}}^{p}}\right)^{\lambda}}}\right)} \\ \frac{\sqrt[q]{\gamma}\, n_{\text{Ą}}^{\lambda}}{\sqrt[q]{\left(1+(\gamma-1)\left(1-n_{\text{Ą}}^{q}\right)\right)^{\lambda}+(\gamma-1)\left(n_{\text{Ą}}^{q}\right)^{2\lambda}}}\, e^{i2\pi\left(\frac{\sqrt[q]{\gamma}\,\varpi_{m_{\text{Ą}}}^{\lambda}}{\sqrt[q]{\left(1+(\gamma-1)\left(1-\varpi_{n_{\text{Ą}}^{q}}\right)\right)^{\lambda}+(\gamma-1)\varpi_{n_{\text{Ą}}^{q}}^{2\lambda}}}\right)} \end{pmatrix}$$

$$= \begin{pmatrix} \sqrt[3]{\frac{(1+(2-1)0.7^3)^2-(1-0.7^3)^2}{(1+(2-1)0.7^3)^2+(2-1)(1-0.7^3)^2}}\, e^{i2\pi\left(\sqrt[3]{\frac{(1+(2-1)0.77^3)^2-(1-0.77^3)^2}{(1+(2-1)0.77^3)^2+(2-1)(1-0.77^3)^2}}\right)} \\ \frac{\sqrt[4]{2}\times 0.69^4}{\sqrt[4]{\left(1+(2-1)(1-0.69^4)\right)^2+(2-1)(0.69^4)^4}}\, e^{i2\pi\left(\frac{\sqrt[4]{2}\times 0.76^4}{\sqrt[4]{\left(1+(2-1)(1-0.76^4)\right)^2+(2-1)(0.76^4)^4}}\right)} \end{pmatrix}$$

$$= \left(\sqrt[3]{\frac{0.96}{2.12}}\, e^{i2\pi\left(\sqrt[3]{\frac{1.826}{2.416}}\right)}, \frac{0.57}{\sqrt[4]{3.5}}\, e^{i2\pi\left(\frac{0.69}{\sqrt[4]{1.79}}\right)}\right)$$

$$= \begin{pmatrix} 0.917e^{i2\pi(0.91)} \\ 0.42e^{i2\pi(0.6)} \end{pmatrix}$$

$$4.\ Ą^{\lambda} = \left(\frac{\sqrt[p]{\gamma} m_{Ą}^{\lambda}}{\sqrt[p]{\left(1+(\gamma-1)\left(1-m_{Ą}^{p}\right)\right)^{\lambda}+(\gamma-1)\left(m_{Ą}^{p}\right)^{2\lambda}}} e^{i2\pi\left(\frac{\sqrt[p]{\gamma}\varpi_{m_{Ą}}^{\lambda}}{\sqrt[p]{\left(1+(\gamma-1)\left(1-\varpi_{m_{Ą}^{p}}\right)\right)^{\lambda}+(\gamma-1)\varpi_{m_{Ą}}^{p\,2\lambda}}}\right)}, \frac{\sqrt[q]{\gamma n_{Ą}^{\lambda}}}{\sqrt[q]{\left(1+(\gamma-1)\left(1-n_{Ą}^{q}\right)\right)^{\lambda}+(\gamma-1)\left(n_{Ą}^{q}\right)^{2\lambda}}} e^{i2\pi\left(\sqrt[q]{\frac{\left(1+(\gamma-1)\varpi_{n_{Ą}^{q}}\right)^{\lambda}-\left(1-\varpi_{n_{Ą}^{q}}\right)^{\lambda}}{\left(1+(\gamma-1)\varpi_{n_{Ą}^{q}}\right)^{\lambda}+(\gamma-1)\left(1-\varpi_{n_{Ą}^{q}}\right)^{\lambda}}}\right)} \right)$$

$$= \left(\frac{\sqrt[3]{2}\times 0.7^{2}}{\sqrt[3]{\left(1+(2-1)(1-0.7^{3})\right)^{2}+(2-1)(1-0.7^{3})^{4}}} e^{i2\pi\left(\frac{\sqrt[3]{2}\times 0.77^{2}}{\sqrt[3]{\left(1+(2-1)(1-0.77^{3})\right)^{2}+(2-1)(1-0.77^{3})^{4}}}\right)}, \sqrt[4]{\frac{(1+(2-1)0.69^{4})^{2}-(1-0.69^{4})^{2}}{(1+(2-1)0.69^{4})^{2}+(2-1)(1-0.69^{4})^{2}}} e^{i2\pi\left(\sqrt[4]{\frac{(1+(2-1)0.76^{4})^{2}-(1-0.76^{4})^{2}}{(1+(2-1)0.76^{4})^{2}+(2-1)(1-0.76^{4})^{2}}}\right)} \right)$$

$$= \left(\frac{0.617}{\sqrt[3]{2.931}} e^{i2\pi\left(\frac{0.74}{\sqrt[3]{2.677}}\right)}, \sqrt[4]{\frac{0.96}{2.12}} e^{i2\pi\left(\sqrt[4]{\frac{1.4}{5.2}}\right)} \right)$$

$$= \begin{pmatrix} 0.431 e^{i2\pi(0.53)} \\ 0.82 e^{i2\pi(0.72)} \end{pmatrix}$$

The Hamacher sum and product are used in the operations suggested in Def. (9) to generalize the Hamacher procedures already in place for CIFSs, CPFSs, and CFSs. As each pair (m, n) corresponds to $p, \mathrm{q} \in \boldsymbol{Z}^{+}$, creating a C*p,q*-$\mathcal{R}$OFN, they define MD, abstention, NMD, and refusal degrees without restrictions. Under the proposed structure, Remark 1 extends these procedures to CPFNs.

Remark *1:* If we assume $p = 3$, $\mathrm{q} = 2$, then complex *p,q*-ROFHO are as follows:

1. $$Ą\oplus B=\left(\begin{array}{c}\left(\frac{m_{Ą}^{3}+m_{B}^{3}-m_{Ą}^{3}m_{B}^{3}-(1-\gamma)m_{Ą}^{3}m_{B}^{3}}{1-(1-\gamma)m_{Ą}^{3}m_{B}^{3}}\right)^{\frac{1}{2}}e^{i2\pi\left(\frac{\varpi_{m_{Ą}^{3}}+\varpi_{m_{B}^{3}}-\varpi_{m_{Ą}^{3}}\varpi_{m_{B}^{3}}-(1-\gamma)\varpi_{m_{Ą}^{3}}\varpi_{m_{B}^{3}}}{1-(1-\gamma)\varpi_{m_{Ą}^{3}}\varpi_{m_{B}^{3}}}\right)^{\frac{1}{2}}}\\ \frac{n_{Ą}n_{B}}{\left(\gamma+(1-\gamma)\left(n_{Ą}^{2}+n_{B}^{2}-n_{Ą}^{2}n_{B}^{2}\right)\right)^{\frac{1}{2}}}e^{i2\pi\left(\frac{\varpi_{n_{Ą}^{2}}\varpi_{n_{B}^{2}}}{\left(\gamma+(1-\gamma)\left(\varpi_{n_{Ą}^{2}}+\varpi_{n_{B}^{2}}-\varpi_{n_{Ą}^{2}}\varpi_{n_{B}^{2}}\right)\right)^{\frac{1}{2}}}\right)}\end{array}\right),\gamma>0$$

2. $$Ą\otimes B=\left(\begin{array}{c}\frac{m_{Ą}m_{B}}{\left(\gamma+(1-\gamma)\left(m_{Ą}^{3}+m_{B}^{3}-m_{Ą}^{3}m_{B}^{3}\right)\right)^{\frac{1}{2}}}e^{i2\pi\left(\frac{\varpi_{m_{Ą}^{3}}\varpi_{m_{B}^{3}}}{\left(\gamma+(1-\gamma)\left(\varpi_{m_{Ą}^{3}}+\varpi_{m_{B}^{3}}-\varpi_{m_{Ą}^{3}}\varpi_{m_{B}^{3}}\right)\right)^{\frac{1}{2}}}\right)}\\ \left(\frac{n_{Ą}^{2}+n_{B}^{2}-n_{Ą}^{2}n_{B}^{2}-(1-\gamma)n_{Ą}^{2}n_{B}^{2}}{1-(1-\gamma)n_{Ą}^{2}n_{B}^{2}}\right)^{\frac{1}{2}}e^{i2\pi\left(\frac{\varpi_{n_{Ą}^{2}}+\varpi_{n_{B}^{2}}-\varpi_{n_{Ą}^{2}}\varpi_{n_{B}^{2}}-(1-\gamma)\varpi_{n_{Ą}^{2}}\varpi_{n_{B}^{2}}}{1-(1-\gamma)\varpi_{n_{Ą}^{2}}\varpi_{n_{B}^{2}}}\right)^{\frac{1}{2}}}\end{array}\right),\gamma>0$$

3. $$\lambda Ą=\left(\begin{array}{c}\left(\frac{\left(1+(\gamma-1)m_{Ą}^{3}\right)^{\lambda}-\left(1-m_{Ą}^{3}\right)^{\lambda}}{\left(1+(\gamma-1)m_{Ą}^{3}\right)^{\lambda}+(\gamma-1)\left(1-m_{Ą}^{3}\right)^{\lambda}}\right)^{\frac{1}{2}}e^{i2\pi\left(\frac{\left(1+(\gamma-1)\varpi_{m_{Ą}^{3}}\right)^{\lambda}-\left(1-\varpi_{m_{Ą}^{3}}\right)^{\lambda}}{\left(1+(\gamma-1)\varpi_{m_{Ą}}^{3}\right)^{\lambda}+(\gamma-1)\left(1-\varpi_{m_{Ą}}^{3}\right)^{\lambda}}\right)^{\frac{1}{2}}}\\ \frac{(\gamma)^{\frac{1}{2}}n_{Ą}^{\lambda}}{\left(\left(1+(\gamma-1)\left(1-n_{Ą}^{2}\right)\right)^{\lambda}+(\gamma-1)\left(n_{Ą}^{2}\right)^{2\lambda}\right)^{\frac{1}{2}}}e^{i2\pi\left(\frac{(\gamma)^{\frac{1}{2}}\varpi_{m_{Ą}}^{\lambda}}{\left(\left(1+(\gamma-1)\left(1-\varpi_{n_{Ą}}^{2}\right)\right)^{\lambda}+(\gamma-1)\left(\varpi_{n_{Ą}}^{2}\right)^{2\lambda}\right)^{\frac{1}{2}}}\right)}\end{array}\right),\gamma>0$$

$$4.\quad Ą^{\lambda} = \left(\begin{array}{c} \dfrac{(\gamma)^{\frac{1}{2}} m_{Ą}^{\lambda}}{\left(\left(1+(\gamma-1)\left(1-m_{Ą}^{3}\right)\right)^{\lambda}+(\gamma-1)\left(m_{Ą}^{3}\right)^{2\lambda}\right)^{\frac{1}{2}}} e^{i2\pi\left(\dfrac{(\gamma)^{\frac{1}{2}} \varpi_{m_{Ą}}^{\lambda}}{\left(\left(1+(\gamma-1)\left(1-\varpi_{m_{Ą}^{3}}\right)\right)^{\lambda}+(\gamma-1)\left(\varpi_{m_{Ą}^{3}}\right)^{2\lambda}\right)^{\frac{1}{2}}}\right)} \\ \dfrac{\sqrt[q]{\gamma n_{Ą}^{\lambda}}}{\left(\left(1+(\gamma-1)\left(1-n_{Ą}^{q}\right)\right)^{\lambda}+(\gamma-1)\left(n_{Ą}^{q}\right)^{2\lambda}\right)^{\frac{1}{2}}} e^{i2\pi\left(\dfrac{\left(1+(\gamma-1)\varpi_{n_{Ą}^{2}}\right)^{\lambda}-\left(1-\varpi_{n_{Ą}^{2}}\right)^{\lambda}}{\left(1+(\gamma-1)\varpi_{n_{Ą}^{2}}\right)^{\lambda}+(\gamma-1)\left(1-\varpi_{n_{Ą}^{2}}\right)^{\lambda}}\right)^{\frac{1}{2}}} \end{array} \right), \gamma > 0$$

3.2 Hamacher Averaging operators in Complex p,q Rung orthopair Fuzzy Sets

The averaging aggregation operators based on Hamacher procedures [18] form the basis of this section. We suggest the C*p,q-ℛ*OFHWA operator using the Hamacher operation suggested in def. (9). The suggested operator is validated using the induction approach and its other characteristics are also examined. Here, $w = (w_1, w_2, w_3 \dots w_n)^{Ŧ}$ are weight vectors where $w_i > 0$ and $\sum_1^n w_{ȷ} = 1$. In indexing sets, the terms ȷ and ƙ are used, where $ȷ, ƙ = 1, 2, 3, \dots$ ł.

Numerical Example:

Consider the following three $C_{p,q}$ -ROFNs:

$$A_1 =< \mu_1 = 0.7 + 0.2i, \nu_1 = 0.3 + 0.1i >$$

$$A_2 =< \mu_2 = 0.5 + 0.3i, \nu_2 = 0.4 + 0.2i >$$

$$A_3 =< \mu_3 = 0.6 + 0.1i, \nu_3 = 0.2 + 0.1i >$$

With corresponding weights, $w = (0.4, 0.3, 0.3)$, using $C_{p,q}$ -ROFHWA operator, the aggregated membership degree and non-membership degree are computed as:

$$\mu_{HWA} = \frac{\sum_{i=1}^{3} w_i \mu_i}{1 + \prod_{i=1}^{3}(1 - w_i . \mu_i)};\ \nu_{HWA} = \frac{\sum_{i=1}^{3} w_i \nu_i}{1 + \prod_{i=1}^{3}(1 - w_i . \nu_i)}$$

Substituting the values (Operations performed on complex numbers component-wise):

$$\mu_{HWA} \approx 0.6 + 0.19i, \qquad \nu_{HWA} \approx 0.3 + 0.14i$$

Thus, the aggregated result is:

$$A_{HWA} =< 0.6 + 0.19i, 0.3 + 0.14i \ >$$

This example demonstrates how the Cp,q-$\mathcal{R}$OFHWA operator effectively integrates individual complex membership and non-membership values under weighted averaging using the Hamacher approach

Definition 10: Consider $\text{Ŧ} = \left(m_i e^{i2\pi\varpi_{m_i}}, n_i e^{i2\pi\varpi_{n_i}}\right)$ is a collection. Then C*p,q*-$\mathcal{R}$OFHWAO is map $T^n \rightharpoonup T$ where C*p,q*-$\mathcal{R}$OHWA $(\text{Ŧ}_1, \text{Ŧ}_2, \text{Ŧ}_3 \ldots \text{Ŧ}_n) = \bigoplus_{j=1}^{l} w_j \text{Ŧ}_j$ by def. 9.

Theorem 1: Let Ŧ_i= (m, n) be a collection. Therefore, the form of C*p,q*-$\mathcal{R}$OFHWAO is,

$$Cp, \mathrm{q} - \mathcal{R}OFHWA(\text{Ŧ}_1, \text{Ŧ}_2, \text{Ŧ}_3 \ldots \text{Ŧ}_n) =$$

$$\left(\begin{array}{c} \sqrt[p]{\frac{\prod_{j=1}^{l}\left(1+(\gamma-1)m_j{}^p\right)^{w_j}-\prod_{j=1}^{l}\left(1-m_j{}^p\right)^{w_j}}{\prod_{j=1}^{l}\left(1+(\gamma-1)m_j{}^p\right)^{w_j}+(\gamma-1)\prod_{j=1}^{l}\left(1-m_j{}^p\right)^{w_j}}}\, e^{i2\pi \sqrt[p]{\frac{\prod_{j=1}^{l}\left(1+(\gamma-1)\varpi_{m_j}{}^p\right)^{w_j}-\prod_{j=1}^{l}\left(1-\varpi_{m_j}{}^p\right)^{w_j}}{\prod_{j=1}^{l}\left(1+(\gamma-1)\varpi_{m_j}{}^p\right)^{w_j}+(\gamma-1)\prod_{j=1}^{l}\left(1-\varpi_{m_j}{}^p\right)^{w_j}}}} \\ \frac{\sqrt[q]{\gamma}\prod_{j=1}^{l} n_j{}^{w_j}}{\sqrt[q]{\prod_{j=1}^{l}\left(1+(\gamma-1)\left(1-n_j{}^q\right)\right)^{w_j}+(\gamma-1)\prod_{j=1}^{l}\left(n_j{}^q\right)^{2w_j}}}\, e^{i2\pi\frac{\sqrt[q]{\gamma}\prod_{j=1}^{l}\varpi_{n_j}{}^{w_j}}{\sqrt[q]{\prod_{j=1}^{l}\left(1+(\gamma-1)\left(1-\varpi_{n_j}{}^q\right)\right)^{w_j}+(\gamma-1)\prod_{j=1}^{l}\left(\varpi_{n_j}{}^q\right)^{2w_j}}}} \end{array}\right) \quad (7)$$

Proof*:* By mathematical induction method. First $\mathrm{l} = 2$, then $w_1\text{Ŧ}_1 \oplus w_2\text{Ŧ}_2$

$$= \left(\begin{array}{c} \sqrt[p]{\frac{(1+(\gamma-1)m_1{}^p)^{w_1}-(1-m_1{}^p)^{w_1}}{(1+(\gamma-1)m_1{}^p)^{w_1}+(\gamma-1)(1-m_1{}^p)^{w_1}}}\, e^{i2\pi\sqrt[p]{\frac{\left(1+(\gamma-1)\varpi_{m_1}{}^p\right)^{w_1}-\left(1-\varpi_{m_1}{}^p\right)^{w_1}}{\left(1+(\gamma-1)\varpi_{m_1}{}^p\right)^{w_1}+(\gamma-1)\left(1-\varpi_{m_1}{}^p\right)^{w_1}}}} \\ \frac{\sqrt[q]{\gamma} n_1^{w_1}}{\sqrt[q]{\left(1+(\gamma-1)(1-n_1{}^q)\right)^{w_1}+(\gamma-1)(n_1{}^q)^{2w_1}}}\, e^{i2\pi\frac{\sqrt[q]{\gamma}\varpi_{n_1}^{w_1}}{\sqrt[q]{\left(1+(\gamma-1)\left(1-\varpi_{n_1}{}^q\right)\right)^{w_1}+(\gamma-1)\left(\varpi_{n_1}{}^q\right)^{2w_1}}}} \end{array}\right)$$

$$\oplus \left(\begin{array}{c} \sqrt[p]{\frac{(1+(\gamma-1)m_2{}^p)^{w_2}-(1-m_2{}^p)^{w_2}}{(1+(\gamma-1)m_2{}^p)^{2}+(\gamma-1)(1-m_2{}^p)^{w_2}}}\, e^{i2\pi\sqrt[p]{\frac{\left(1+(\gamma-1)\varpi_{m_2}{}^p\right)^{w_2}-\left(1-\varpi_{m_2}{}^p\right)^{w_2}}{\left(1+(\gamma-1)\varpi_{m_2}{}^p\right)^{w_2}+(\gamma-1)\left(1-\varpi_{m_2}{}^p\right)^{w_2}}}} \\ \frac{\sqrt[q]{\gamma} n_2^{w_2}}{\sqrt[q]{\left(1+(\gamma-1)(1-n_2{}^q)\right)^{w_2}+(\gamma-1)(n_2{}^q)^{2w_2}}}\, e^{i2\pi\frac{\sqrt[q]{\gamma}\varpi_{n_2}^{w_2}}{\sqrt[q]{\left(1+(\gamma-1)\left(1-\varpi_{n_2}{}^q\right)\right)^{w_2}+(\gamma-1)\left(\varpi_{n_2}{}^q\right)^{2w_2}}}} \end{array}\right)$$

$$w_1\text{Ŧ}_1 \oplus w_2\text{Ŧ}_2$$

$$= \begin{pmatrix} \sqrt[p]{\frac{\prod_{j=1}^{2}\left(1+(\gamma-1)m_j^{p}\right)^{w_j}-\prod_{j=1}^{2}\left(1-m_j^{p}\right)^{w_j}}{\prod_{j=1}^{2}\left(1+(\gamma-1)m_j^{p}\right)^{w_j}+(\gamma-1)\prod_{j=1}^{2}\left(1-m_j^{p}\right)^{w_j}}}\, e^{i2\pi\sqrt[p]{\frac{\prod_{j=1}^{2}\left(1+(\gamma-1)\varpi_{m_j}^{p}\right)^{w_j}-\prod_{j=1}^{2}\left(1-\varpi_{m_j}^{p}\right)^{w_j}}{\prod_{j=1}^{2}\left(1+(\gamma-1)\varpi_{m_j}^{p}\right)^{w_j}+(\gamma-1)\prod_{j=1}^{2}\left(1-\varpi_{m_j}^{p}\right)^{w_j}}}} \\ \frac{\sqrt[q]{\gamma}\prod_{j=1}^{2} n_j^{w_j}}{\sqrt[q]{\prod_{j=1}^{2}\left(1+(\gamma-1)\left(1-n_j^{q}\right)\right)^{w_j}+(\gamma-1)\prod_{j=1}^{2}\left(n_j^{q}\right)^{2w_j}}}\, e^{i2\pi\frac{\sqrt[q]{\gamma}\prod_{j=1}^{2}\varpi_{n_j}^{w_j}}{\sqrt[q]{\prod_{j=1}^{2}\left(1+(\gamma-1)\left(1-\varpi_{n_j}^{q}\right)\right)^{w_j}+(\gamma-1)\prod_{j=1}^{2}\left(\varpi_{n_j}^{q}\right)^{2w_j}}}} \end{pmatrix}$$

Now suppose this result $l = k$ then

$$Cp,q-\mathcal{R}OFHWA(\text{Ŧ}_1,\text{Ŧ}_2,\text{Ŧ}_3\ldots\text{Ŧ}_k)$$

$$= \begin{pmatrix} \sqrt[p]{\frac{\prod_{j=1}^{k}\left(1+(\gamma-1)m_j^{p}\right)^{w_j}-\prod_{j=1}^{k}\left(1-m_j^{p}\right)^{w_j}}{\prod_{j=1}^{k}\left(1+(\gamma-1)m_j^{p}\right)^{w_j}+(\gamma-1)\prod_{j=1}^{k}\left(1-m_j^{p}\right)^{w_j}}}\, e^{i2\pi\sqrt[p]{\frac{\prod_{j=1}^{k}\left(1+(\gamma-1)\varpi_{m_j}^{p}\right)^{w_j}-\prod_{j=1}^{k}\left(1-\varpi_{m_j}^{p}\right)^{w_j}}{\prod_{j=1}^{k}\left(1+(\gamma-1)\varpi_{m_j}^{p}\right)^{w_j}+(\gamma-1)\prod_{j=1}^{k}\left(1-\varpi_{m_j}^{p}\right)^{w_j}}}} \\ \frac{\sqrt[q]{\gamma}\prod_{j=1}^{k} n_j^{w_j}}{\sqrt[q]{\prod_{j=1}^{k}\left(1+(\gamma-1)\left(1-n_j^{q}\right)\right)^{w_j}+(\gamma-1)\prod_{j=1}^{k}\left(n_j^{q}\right)^{2w_j}}}\, e^{i2\pi\frac{\sqrt[q]{\gamma}\prod_{j=1}^{k}\varpi_{n_j}^{w_j}}{\sqrt[q]{\prod_{j=1}^{k}\left(1+(\gamma-1)\left(1-\varpi_{n_j}^{q}\right)\right)^{w_j}+(\gamma-1)\prod_{j=1}^{k}\left(\varpi_{n_j}^{q}\right)^{2w_j}}}} \end{pmatrix}$$

Now $l = k + 1$

$Cp,q\text{-}\mathcal{R}\text{OFHWA}\ (\text{Ŧ}_1,\text{Ŧ}_2,\text{Ŧ}_3\ldots\text{Ŧ}_k,\text{Ŧ}_{k+1}) = Cp,q\text{-}\mathcal{R}\text{OFHWA}\ (\text{Ŧ}_1,\text{Ŧ}_2,\text{Ŧ}_3\ldots\text{Ŧ}_k)\oplus\text{Ŧ}_{k+1}$

$$= \begin{pmatrix} \sqrt[p]{\frac{\prod_{j=1}^{k}\left(1+(\gamma-1)m_j^{p}\right)^{w_j}-\prod_{j=1}^{k}\left(1-m_j^{p}\right)^{w_j}}{\prod_{j=1}^{k}\left(1+(\gamma-1)m_j^{p}\right)^{w_j}+(\gamma-1)\prod_{j=1}^{k}\left(1-m_j^{p}\right)^{w_j}}}\, e^{i2\pi\sqrt[p]{\frac{\prod_{j=1}^{k}\left(1+(\gamma-1)\varpi_{m_j}^{p}\right)^{w_j}-\prod_{j=1}^{k}\left(1-\varpi_{m_j}^{p}\right)^{w_j}}{\prod_{j=1}^{k}\left(1+(\gamma-1)\varpi_{m_j}^{p}\right)^{w_j}+(\gamma-1)\prod_{j=1}^{k}\left(1-\varpi_{m_j}^{p}\right)^{w_j}}}} \\ \frac{\sqrt[q]{\gamma}\prod_{j=1}^{k} n_j^{w_j}}{\sqrt[q]{\prod_{j=1}^{k}\left(1+(\gamma-1)\left(1-n_j^{q}\right)\right)^{w_j}+(\gamma-1)\prod_{j=1}^{k}\left(n_j^{q}\right)^{2w_j}}}\, e^{i2\pi\frac{\sqrt[q]{\gamma}\prod_{j=1}^{k}\varpi_{n_j}^{w_j}}{\sqrt[q]{\prod_{j=1}^{k}\left(1+(\gamma-1)\left(1-\varpi_{n_j}^{q}\right)\right)^{w_j}+(\gamma-1)\prod_{j=1}^{k}\left(\varpi_{n_j}^{q}\right)^{2w_j}}}} \end{pmatrix}$$

$$\oplus \begin{pmatrix} \sqrt[p]{\frac{(1+(\gamma-1)m_{k+1}^{p})^{w_{k+1}}-(1-m_{k+1}^{p})^{w_{k+1}}}{(1+(\gamma-1)m_{k+1}^{p})^{w_{k+1}}+(\gamma-1)(1-m_{k+1}^{p})^{w_{k+1}}}}\, e^{i2\pi\sqrt[p]{\frac{(1+(\gamma-1)\varpi_{m_{k+1}}^{p})^{w_{k+1}}-(1-\varpi_{m_{k+1}}^{p})^{w_{k+1}}}{(1+(\gamma-1)\varpi_{m_{k+1}}^{p})^{w_{k+1}}+(\gamma-1)(1-\varpi_{m_{k+1}}^{q})^{w_{k+1}}}}} \\ \frac{\sqrt[q]{\gamma}n_{k+1}^{w_{k+1}}}{\sqrt[q]{\left(1+(\gamma-1)(1-n_{k+1}^{q})\right)^{w_{k+1}}+(\gamma-1)(n_{k+1}^{q})^{2w_{k+1}}}}\, e^{i2\pi\frac{\sqrt[q]{\gamma}\varpi_{n_{k+1}}^{w_{k+1}}}{\sqrt[q]{\left(1+(\gamma-1)(1-\varpi_{n_{k+1}}^{q})\right)^{w_{k+1}}+(\gamma-1)(\varpi_{n_{k+1}}^{q})^{2w_{k+1}}}}} \end{pmatrix}$$

C*p,q*-$\mathcal{R}$OFHWA$(Ŧ_1, Ŧ_2, Ŧ_3 \dots Ŧ_{k+1}) =$

$$\left(\sqrt[p]{\frac{\prod_{ȷ=1}^{k+1}\left(1+(\gamma-1)m_ȷ{}^p\right)^{w_ȷ}-\prod_{ȷ=1}^{k+1}\left(1-m_ȷ{}^p\right)^{w_ȷ}}{\prod_{ȷ=1}^{k+1}\left(1+(\gamma-1)m_ȷ{}^p\right)^{w_ȷ}+(\gamma-1)\prod_{ȷ=1}^{k+1}\left(1-m_ȷ{}^p\right)^{w_ȷ}}} e^{i2\pi \sqrt[p]{\frac{\prod_{ȷ=1}^{k+1}\left(1+(\gamma-1)\varpi_{m_ȷ}{}^p\right)^{w_ȷ}-\prod_{ȷ=1}^{k+1}\left(1-\varpi_{m_ȷ}{}^p\right)^{w_ȷ}}{\prod_{ȷ=1}^{k+1}\left(1+(\gamma-1)\varpi_{m_ȷ}{}^p\right)^{w_ȷ}+(\gamma-1)\prod_{ȷ=1}^{k+1}\left(1-\varpi_{m_ȷ}{}^p\right)^{w_ȷ}}}}, \frac{\sqrt[q]{\gamma}\prod_{ȷ=1}^{k+1} n_ȷ{}^{w_ȷ}}{\sqrt[q]{\prod_{ȷ=1}^{k+1}\left(1+(\gamma-1)\left(1-n_ȷ{}^q\right)\right)^{w_ȷ}+(\gamma-1)\prod_{ȷ=1}^{k+1}\left(n_ȷ{}^q\right)^{2w_ȷ}}} e^{i2\pi \frac{\sqrt[q]{\gamma}\prod_{ȷ=1}^{k+1}\varpi_{n_ȷ}{}^{w_ȷ}}{\sqrt[q]{\prod_{ȷ=1}^{k+1}\left(1+(\gamma-1)\left(1-\varpi_{n_ȷ}{}^q\right)\right)^{w_ȷ}+(\gamma-1)\prod_{ȷ=1}^{k+1}\left(\varpi_{n_ȷ}{}^q\right)^{2w_ȷ}}}} \right)$$

The result is valid for all values of l *and* for $l = k + 1$.

In the following Theorem, we now list several fundamental characteristics of the suggested C*p,q*-$\mathcal{R}$OFHWA operator.

Theorem 2: The following properties are satisfied by the Hamacher aggregation operator of C*p,q*-$\mathcal{R}$OFNs

i. (Idempotency) If $Ŧ_ȷ = Ŧ = \left(m_i e^{i2\pi\varpi_{m_i}}, n_i e^{i2\pi\varpi_{n_i}}\right) = \left(m e^{i2\pi\varpi_m}, n e^{i2\pi\varpi_n}\right)\ \forall\, ȷ = 1,2,3 \dots l$. Then C*p,q*-$\mathcal{R}$OFHWA $(Ŧ_1, Ŧ_2, Ŧ_3 \dots Ŧ_n) = Ŧ$.
ii. (Boundedness) If $Ŧ^- = \left(\min_ȷ m_ȷ, \max_ȷ n_ȷ\right)$ and $Ŧ^+ = \left(\min_ȷ m_ȷ, \max_ȷ n_ȷ\right)$. Then
$$Ŧ^- \leq \text{C}p,q\text{-}\mathcal{R}\text{OFHWA}\ (Ŧ_1, Ŧ_2, Ŧ_3 \dots Ŧ_n) \leq Ŧ^+.$$
iii. (Monotonically) Let $Ŧ_ȷ$ and $P_ȷ$ be two C*p,q*-$\mathcal{R}$OFNs such that $Ŧ_ȷ \leq P_ȷ \forall ȷ$.Then C*p,q*-$\mathcal{R}$OFHWA $(Ŧ_1, Ŧ_2, Ŧ_3 \dots Ŧ_n) \leq$ C*p,q*-$\mathcal{R}$OFHWA $(P_1, P_2, P_3 \dots P_n)$

This is demonstrable in an analogous way. C*p,q*-$\mathcal{R}$OFn is the sole one that is weighed by the C*p,q*-$\mathcal{R}$OFHWA aggregation operator. There are situations in which the C*p,q*-$\mathcal{R}$OFN's ordered position is important in MADM problems. In those cases, the idea of ordered weighted averaging operators is important, and the C*p,q*-$\mathcal{R}$OFHOWAO will be suggested as a solution.

Definition 11: Let $Ŧ = \left(m_i e^{i2\pi\varpi_{m_i}}, n_i e^{i2\pi\varpi_{n_i}}\right)$ is a collection. Then C*p,q*-$\mathcal{R}$OFHWAO is a map $T^n \rightharpoonup T$ and C*p,q*-$\mathcal{R}$OFHWA $(Ŧ_1, Ŧ_2, Ŧ_3 \dots Ŧ_n) = \bigoplus_{ȷ=1}^{l} w_ȷ Ŧ_{\sigma(ȷ)}$ where $\sigma(ȷ)$ is such that $Ŧ_{\sigma(ȷ-1)} \geq Ŧ_{\sigma(ȷ)} \forall ȷ$.

Theorem 3: Let $Ŧ = \left(m_i e^{i2\pi\varpi_{m_i}}, n_i e^{i2\pi\varpi_{n_i}}\right)$ = is a collection. Then form of C*p,q*-$\mathcal{R}$OFHWAO is defined as:

$$\text{C}p,q\text{-}\mathcal{R}\text{OFHWA}\ (Ŧ_1, Ŧ_2, Ŧ_3 \dots Ŧ_n) = \bigoplus_{ȷ=1}^{l} w_ȷ Ŧ_{\sigma(ȷ)}$$

$$= \left(\begin{array}{c} \sqrt[p]{\frac{\prod_{j=1}^{l}\left(1+(\gamma-1)m_{\sigma(j)}{}^{p}\right)^{w_j}-\prod_{j=1}^{l}\left(1-m_{\sigma(j)}{}^{p}\right)^{w_j}}{\prod_{j=1}^{l}\left(1+(\gamma-1)m_{\sigma(j)}{}^{p}\right)^{w_j}+(\gamma-1)\prod_{j=1}^{l}\left(1-m_{\sigma(j)}{}^{p}\right)^{w_j}}}\, e^{i2\pi \sqrt[p]{\frac{\prod_{j=1}^{l}\left(1+(\gamma-1)\varpi m_{\sigma(j)}{}^{p}\right)^{w_j}-\prod_{j=1}^{l}\left(1-\varpi m_{\sigma(j)}{}^{p}\right)^{w_j}}{\prod_{j=1}^{l}\left(1+(\gamma-1)\varpi m_{\sigma(j)}{}^{p}\right)^{w_j}+(\gamma-1)\prod_{j=1}^{l}\left(1-\varpi m_{\sigma(j)}{}^{p}\right)^{w_j}}}} \\ \frac{\sqrt[q]{\gamma}\prod_{j=1}^{l} n_{\sigma(j)}^{w_j}}{\sqrt[q]{\prod_{j=1}^{l}\left(1+(\gamma-1)\left(1-n_{\sigma(j)}{}^{q}\right)\right)^{w_j}+(\gamma-1)\prod_{j=1}^{l}\left(n_{\sigma(j)}{}^{q}\right)^{2w_j}}}\, e^{i2\pi \frac{\sqrt[q]{\gamma}\prod_{j=1}^{l}\varpi n_{\sigma(j)}^{w_j}}{\sqrt[q]{\prod_{j=1}^{l}\left(1+(\gamma-1)\left(1-\varpi n_{\sigma(j)}{}^{q}\right)\right)^{w_j}+(\gamma-1)\prod_{j=1}^{l}\left(\varpi n_{\sigma(j)}{}^{q}\right)^{2w_j}}}} \end{array} \right) \quad (8)$$

Remark 2: The C*p,q*-$\mathcal{R}$OFHOWAO described in Eq. (8) satisfies Theorem 2 requirements for idempotency, monotonicity, and boundedness. Whereas the C*p,q*-$\mathcal{R}$OFHOWA operator Eq. (7) directly weights ordered C*p,q* -rung orthopair fuzzy arguments, the C*p,q*-$\mathcal{R}$OFHWA operator (Eq. 8) weighs them. To close this gap, we suggest a hybrid operator.

Definition 12: Let $\text{Ŧ} = \left(m_i e^{i2\pi\varpi_{m_i}}, n_i e^{i2\pi\varpi_{n_i}}\right)$ be a collection. So, C*p,q*-$\mathcal{R}$OFHHAO is a map $T^n \rightharpoonup T$ such that

$$\text{C}p,q\text{-}\mathcal{R}\text{OFHHA}\left(\text{Ŧ}_1, \text{Ŧ}_2, \text{Ŧ}_3 \ldots \text{Ŧ}_n\right) = \bigoplus_{j=1}^{l} w_j \dot{\text{Ŧ}}_{\sigma(j)}$$

Where $\dot{\text{Ŧ}}_{\sigma(j)}$ is the jth largest of the TSFN $\dot{\text{Ŧ}}_j = |\omega_j \text{Ŧ}_j$ with ω_j as the weight vector of C*p,q*-$\mathcal{R}$OF arguments Ŧ_j where $\omega_j \in [0, 1]$ and $\sum_1^n \omega_j = 1$ and l is the balancing coefficient.

Theorem 4: Let $\text{Ŧ} = \left(m_i e^{i2\pi\varpi_{m_i}}, n_i e^{i2\pi\varpi_{n_i}}\right)$ be a collection. The form of C*p,q*-$\mathcal{R}$OFHHAO is:

C*p,q*-$\mathcal{R}$OFHHA$\left(\text{Ŧ}_1, \text{Ŧ}_2, \text{Ŧ}_3 \ldots \text{Ŧ}_n\right) =$

$$\left(\begin{array}{c} \sqrt[p]{\frac{\prod_{j=1}^{l}\left(1+(\gamma-1)\dot{m}_{\sigma(j)}{}^{p}\right)^{w_j}-\prod_{j=1}^{l}\left(1-\dot{m}_{\sigma(j)}{}^{p}\right)^{w_j}}{\prod_{j=1}^{l}\left(1+(\gamma-1)\dot{m}_{\sigma(j)}{}^{p}\right)^{w_j}+(\gamma-1)\prod_{j=1}^{l}\left(1-\dot{m}_{\sigma(j)}{}^{p}\right)^{w_j}}}\, e^{i2\pi \sqrt[p]{\frac{\prod_{j=1}^{l}\left(1+(\gamma-1)\varpi \dot{m}_{\sigma(j)}{}^{p}\right)^{w_j}-\prod_{j=1}^{l}\left(1-\varpi \dot{m}_{\sigma(j)}{}^{p}\right)^{w_j}}{\prod_{j=1}^{l}\left(1+(\gamma-1)\varpi \dot{m}_{\sigma(j)}{}^{p}\right)^{w_j}+(\gamma-1)\prod_{j=1}^{l}\left(1-\varpi \dot{m}_{\sigma(j)}{}^{p}\right)^{w_j}}}} \\ \frac{\sqrt[q]{\gamma}\prod_{j=1}^{l} \dot{n}_{\sigma(j)}^{w_j}}{\sqrt[q]{\prod_{j=1}^{l}\left(1+(\gamma-1)\left(1-\dot{n}_{\sigma(j)}{}^{q}\right)\right)^{w_j}+(\gamma-1)\prod_{j=1}^{l}\left(\dot{n}_{\sigma(j)}{}^{q}\right)^{2w_j}}}\, e^{i2\pi \frac{\sqrt[q]{\gamma}\prod_{j=1}^{l}\varpi \dot{n}_{\sigma(j)}^{w_j}}{\sqrt[q]{\prod_{j=1}^{l}\left(1+(\gamma-1)\left(1-\varpi \dot{n}_{\sigma(j)}{}^{q}\right)\right)^{w_j}+(\gamma-1)\prod_{j=1}^{l}\left(\varpi \dot{n}_{\sigma(j)}{}^{q}\right)^{2w_j}}}} \end{array} \right) \quad (9)$$

Remark *3:* The Eq. (9) will move to Cp,*q*-$\mathcal{R}$OFHWAO if we are taking $w_j = \left(\frac{1}{l}, \frac{1}{l}, \frac{1}{l} \ldots \frac{1}{l}\right)^T$ while it moves to C*p,q*-$\mathcal{R}$OFHOWAO if we are taking $\omega_j = \left(\frac{1}{l}, \frac{1}{l}, \frac{1}{l} \ldots \frac{1}{l}\right)^T$.

3.3 Complex p,q-Rung Orthopair Fuzzy Hamacher Geometric Operators

In this section, HOs serve as the foundation for geometric aggregation operators. In Def. (9), The C*p,q*-$\mathcal{R}$OFHWGO, which is based on the Hamacher operation. Validation of the proposed operator is done by the induction approach. Additionally, certain other characteristics of the C*p,q*-$\mathcal{R}$OFHWG operator are examined.

Definition 13: Let $Ŧ = \left(m_i e^{i2\pi\varpi_{m_i}}, n_i e^{i2\pi\varpi_{n_i}}\right)$ be a collection. Then C*p,q*-$\mathcal{R}$OFHWGO with mapping $T^n \rightharpoonup T$:

$$\text{C}p,q\text{-}\mathcal{R}\text{OFHWG}(Ŧ_1, Ŧ_2, Ŧ_3 \dots Ŧ_n) = \bigoplus_{\mathrm{j}=1}^{\mathrm{l}} \mathrm{w_j} Ŧ_\mathrm{j} =$$

$$\left(\begin{array}{c} \dfrac{\sqrt[p]{\gamma}\prod_{\mathrm{j}=1}^{\mathrm{l}} m_\mathrm{j}^{\mathrm{w_j}}}{\sqrt[p]{\prod_{\mathrm{j}=1}^{\mathrm{l}}\left(1+(\gamma-1)\left(1-m_\mathrm{j}{}^p\right)\right)^{\mathrm{w_j}}+(\gamma-1)\prod_{\mathrm{j}=1}^{\mathrm{l}}\left(m_\mathrm{j}{}^p\right)^{2\mathrm{w_j}}}} e^{i2\pi \frac{\sqrt[p]{\gamma}\prod_{\mathrm{j}=1}^{\mathrm{l}} \varpi_{m_\mathrm{j}}^{\mathrm{w_j}}}{\sqrt[p]{\prod_{\mathrm{j}=1}^{\mathrm{l}}\left(1+(\gamma-1)\left(1-\varpi_{m_\mathrm{j}}{}^p\right)\right)^{\mathrm{w_j}}+(\gamma-1)\prod_{\mathrm{j}=1}^{\mathrm{l}}\left(\varpi_{m_\mathrm{j}}{}^p\right)^{2\mathrm{w_j}}}}} \\ \sqrt[q]{\dfrac{\prod_{\mathrm{j}=1}^{\mathrm{l}}\left(1+(\gamma-1)n_\mathrm{j}{}^q\right)^{\mathrm{w_j}}-\prod_{\mathrm{j}=1}^{\mathrm{l}}\left(1-n_\mathrm{j}{}^q\right)^{\mathrm{w_j}}}{\prod_{\mathrm{j}=1}^{\mathrm{l}}\left(1+(\gamma-1)n_\mathrm{j}{}^q\right)^{\mathrm{w_j}}+(\gamma-1)\prod_{\mathrm{j}=1}^{\mathrm{l}}\left(1-n_\mathrm{j}{}^q\right)^{\mathrm{w_j}}}} e^{i2\pi \sqrt[q]{\frac{\prod_{\mathrm{j}=1}^{\mathrm{l}}\left(1+(\gamma-1)\varpi_{n_\mathrm{j}}{}^q\right)^{\mathrm{w_j}}-\prod_{\mathrm{j}=1}^{\mathrm{l}}\left(1-\varpi_{n_\mathrm{j}}{}^q\right)^{\mathrm{w_j}}}{\prod_{\mathrm{j}=1}^{\mathrm{l}}\left(1+(\gamma-1)\varpi_{n_\mathrm{j}}{}^q\right)^{\mathrm{w_j}}+(\gamma-1)\prod_{\mathrm{j}=1}^{\mathrm{l}}\left(1-\varpi_{n_\mathrm{j}}{}^q\right)^{\mathrm{w_j}}}}} \end{array}\right) \quad (10)$$

However, we now propose this as a result, using Def. 9.

Theorem 5: Let $Ŧ = \left(m_i e^{i2\pi\varpi_{m_i}}, n_i e^{i2\pi\varpi_{n_i}}\right)$ be a collection. The form of C*p,q*-$\mathcal{R}$OFHWGO is

$$TSFHWG(Ŧ_1, Ŧ_2, Ŧ_3 \dots Ŧ_n) = \bigotimes_{\mathrm{j}=1}^{\mathrm{l}} Ŧ_\mathrm{j}^{\mathrm{w_j}}$$

Proof*:* By demonstrating the outcome using mathematical induction.

For $\mathrm{l} = 2$

$$\mathrm{w}_1 Ŧ_1 \oplus \mathrm{w}_2 Ŧ_2 =$$

$$= \left(\begin{array}{c} \dfrac{\sqrt[p]{\gamma} n_1^{\mathrm{w}_1}}{\sqrt[p]{\left(1+(\gamma-1)\left(1-n_1{}^p\right)\right)^{\mathrm{w}_1}+(\gamma-1)\left(n_1{}^p\right)^{2\mathrm{w}_1}}} e^{i2\pi \frac{\sqrt[p]{\gamma}\varpi_{n_1}^{\mathrm{w}_1}}{\sqrt[p]{\left(1+(\gamma-1)\left(1-\varpi_{n_1}{}^p\right)\right)^{\mathrm{w}_1}+(\gamma-1)\left(\varpi_{n_1}{}^p\right)^{2\mathrm{w}_1}}}} \\ \sqrt[q]{\dfrac{\left(1+(\gamma-1)m_1{}^q\right)^{\mathrm{w}_1}-\left(1-m_1{}^q\right)^{\mathrm{w}_1}}{\left(1+(\gamma-1)m_1{}^q\right)^{\mathrm{w}_1}+(\gamma-1)\left(1-m_1{}^q\right)^{\mathrm{w}_1}}} e^{i2\pi \sqrt[q]{\frac{\left(1+(\gamma-1)\varpi_{m_1}{}^q\right)^{\mathrm{w}_1}-\left(1-\varpi_{m_1}{}^q\right)^{\mathrm{w}_1}}{\left(1+(\gamma-1)\varpi_{m_1}{}^q\right)^{\mathrm{w}_1}+(\gamma-1)\left(1-\varpi_{m_1}{}^q\right)^{\mathrm{w}_1}}}} \end{array}\right)$$

$$\oplus\left(\frac{\sqrt[p]{\gamma}\,n_2^{w_2}}{\sqrt[p]{\left(1+(\gamma-1)(1-n_2^{\,p})\right)^{w_2}+(\gamma-1)(n_2^{\,p})^{2w_2}}}e^{i2\pi\frac{\sqrt[p]{\gamma}\,\varpi_{n_2}^{w_2}}{\sqrt[p]{\left(1+(\gamma-1)(1-\varpi_{n_2}^{\,p})\right)^{w_2}+(\gamma-1)(\varpi_{n_2}^{\,p})^{2w_2}}}},\ \sqrt[q]{\frac{(1+(\gamma-1)m_2^{\,q})^{w_2}-(1-m_2^{\,q})^{w_2}}{(1+(\gamma-1)m_2^{\,q})^{2}+(\gamma-1)(1-m_2^{\,q})^{w_2}}}\,e^{i2\pi\sqrt[q]{\frac{(1+(\gamma-1)\varpi_{m_2}^{\,q})^{w_2}-(1-\varpi_{m_2}^{\,q})^{w_2}}{(1+(\gamma-1)\varpi_{m_2}^{\,q})^{w_2}+(\gamma-1)(1-\varpi_{m_2}^{\,q})^{w_2}}}}\right)$$

$$w_1Ŧ_1\oplus w_2Ŧ_2$$

$$=\left(\frac{\sqrt[p]{\gamma}\prod_{j=1}^{2}m_j^{w_j}}{\sqrt[p]{\prod_{j=1}^{2}\left(1+(\gamma-1)(1-m_j^{\,p})\right)^{w_j}+(\gamma-1)\prod_{j=1}^{2}(m_j^{\,p})^{2w_j}}}e^{i2\pi\frac{\sqrt[p]{\gamma}\prod_{j=1}^{2}\varpi_{m_j}^{w_j}}{\sqrt[p]{\prod_{j=1}^{2}\left(1+(\gamma-1)(1-\varpi_{m_j}^{\,p})\right)^{w_j}+(\gamma-1)\prod_{j=1}^{2}(\varpi_{m_j}^{\,p})^{2w_j}}}},\ \sqrt[q]{\frac{\prod_{j=1}^{2}(1+(\gamma-1)n_j^{\,q})^{w_j}-\prod_{j=1}^{2}(1-n_j^{\,q})^{w_j}}{\prod_{j=1}^{2}(1+(\gamma-1)n_j^{\,q})^{w_j}+(\gamma-1)\prod_{j=1}^{2}(1-n_j^{\,q})^{w_j}}}\,e^{i2\pi\sqrt[q]{\frac{\prod_{j=1}^{2}(1+(\gamma-1)\varpi_{n_j}^{\,q})^{w_j}-\prod_{j=1}^{2}(1-\varpi_{n_j}^{\,q})^{w_j}}{\prod_{j=1}^{2}(1+(\gamma-1)\varpi_{n_j}^{\,q})^{w_j}+(\gamma-1)\prod_{j=1}^{2}(1-\varpi_{n_j}^{\,q})^{w_j}}}}\right)$$

For $l = 2$, the outcome in Eq. (10) is valid. If it is true for $l = ꝁ$, then.

C*p,q*-$\mathcal{R}$OFHWA$(Ŧ_1, Ŧ_2, Ŧ_3 \ldots Ŧ_k) =$

$$\left(\frac{\sqrt[p]{\gamma}\prod_{j=1}^{ꝁ}m_j^{w_j}}{\sqrt[p]{\prod_{j=1}^{ꝁ}\left(1+(\gamma-1)(1-m^{p})\right)^{w_j}+(\gamma-1)\prod_{j=1}^{ꝁ}(m_j^{\,p})^{2w_j}}}e^{i2\pi\frac{\sqrt[p]{\gamma}\prod_{j=1}^{ꝁ}\varpi_{m_j}^{w_j}}{\sqrt[p]{\prod_{j=1}^{ꝁ}\left(1+(\gamma-1)(1-\varpi_{m_j}^{\,p})\right)^{w_j}+(\gamma-1)\prod_{j=1}^{ꝁ}(\varpi_{m_j}^{\,p})^{2w_j}}}},\ \sqrt[q]{\frac{\prod_{j=1}^{ꝁ}(1+(\gamma-1)n_j^{\,q})^{w_j}-\prod_{j=1}^{ꝁ}(1-n_j^{\,q})^{w_j}}{\prod_{j=1}^{ꝁ}(1+(\gamma-1)n_j^{\,q})^{w_j}+(\gamma-1)\prod_{j=1}^{ꝁ}(1-n_j^{\,q})^{w_j}}}\,e^{i2\pi\sqrt[q]{\frac{\prod_{j=1}^{ꝁ}(1+(\gamma-1)\varpi_{n_j}^{\,q})^{w_j}-\prod_{j=1}^{ꝁ}(1-\varpi_{n_j}^{\,q})^{w_j}}{\prod_{j=1}^{ꝁ}(1+(\gamma-1)\varpi_{n_j}^{\,q})^{w_j}+(\gamma-1)\prod_{j=1}^{ꝁ}(1-\varpi_{n_j}^{\,q})^{w_j}}}}\right)$$

Now, $l = ꝁ + 1$

C*p,q*-$\mathcal{R}$OFHWG $(Ŧ_1, Ŧ_2, Ŧ_3 \ldots Ŧ_{ꝁ}, Ŧ_{ꝁ+1})=$ C*p,q*-$\mathcal{R}$OFHWA$(Ŧ_1, Ŧ_2, Ŧ_3 \ldots Ŧ_{ꝁ})\oplus Ŧ_{ꝁ+1}$

$$\left(\begin{array}{c}\frac{\sqrt[p]{\gamma}\prod_{\text{j}=1}^{\text{k}} m_{\text{j}}^{w_{\text{j}}}}{\sqrt[p]{\prod_{\text{j}=1}^{\text{k}}\left(1+(\gamma-1)\left(1-m_{\text{j}}{}^{p}\right)\right)^{w_{\text{j}}}+(\gamma-1)\prod_{\text{j}=1}^{\text{k}}\left(m_{\text{j}}{}^{p}\right)^{2w_{\text{j}}}}} e^{i2\pi\frac{\sqrt[p]{\gamma}\prod_{\text{j}=1}^{\text{k}}\varpi_{m_{\text{j}}}{}^{w_{\text{j}}}}{\sqrt[p]{\prod_{\text{j}=1}^{\text{k}}\left(1+(\gamma-1)\left(1-\varpi_{m_{\text{j}}}{}^{p}\right)\right)^{w_{\text{j}}}+(\gamma-1)\prod_{\text{j}=1}^{\text{k}}\left(\varpi_{m_{\text{j}}}{}^{p}\right)^{2w_{\text{j}}}}}} \\ \sqrt[q]{\frac{\prod_{\text{j}=1}^{\text{k}}\left(1+(\gamma-1)n_{\text{j}}{}^{q}\right)^{w_{\text{j}}}-\prod_{\text{j}=1}^{\text{k}}\left(1-n_{\text{j}}{}^{q}\right)^{w_{j}}}{\prod_{\text{j}=1}^{\text{k}}\left(1+(\gamma-1)n_{\text{j}}{}^{q}\right)^{w_{\text{j}}}+(\gamma-1)\prod_{\text{j}=1}^{\text{k}}\left(1-n_{\text{j}}{}^{q}\right)^{w_{\text{j}}}}} e^{i2\pi\sqrt[q]{\frac{\prod_{\text{j}=1}^{\text{k}}\left(1+(\gamma-1)\varpi_{n_{\text{j}}}{}^{q}\right)^{w_{\text{j}}}-\prod_{\text{j}=1}^{\text{k}}\left(1-\varpi_{n_{\text{j}}}{}^{q}\right)^{w_{\text{j}}}}{\prod_{\text{j}=1}^{\text{k}}\left(1+(\gamma-1)\varpi_{n_{\text{j}}}{}^{q}\right)^{w_{\text{j}}}+(\gamma-1)\prod_{\text{j}=1}^{\text{k}}\left(1-\varpi_{n_{\text{j}}}{}^{q}\right)^{w_{\text{j}}}}}}\end{array}\right)$$

$$\oplus\left(\begin{array}{c}\frac{\sqrt[p]{\gamma} m_{\text{k}+1}^{w_{\text{k}+1}}}{\sqrt[p]{\left(1+(\gamma-1)\left(1-m_{\text{k}+1}{}^{p}\right)\right)^{w_{\text{k}+1}}+(\gamma-1)\left(m_{\text{k}+1}{}^{p}\right)^{2w_{\text{k}+1}}}} e^{i2\pi\frac{\sqrt[p]{\gamma}\varpi_{m_{\text{k}+1}}{}^{w_{\text{k}+1}}}{\sqrt[p]{\left(1+(\gamma-1)\left(1-\varpi_{m_{\text{k}+1}}{}^{p}\right)\right)^{w_{\text{k}+1}}+(\gamma-1)\left(\varpi_{m_{\text{k}+1}}{}^{p}\right)^{2w_{\text{k}+1}}}}} \\ \sqrt[p,q]{\frac{\left(1+(\gamma-1)n_{\text{k}+1}{}^{q}\right)^{w_{\text{k}+1}}-\left(1-n_{\text{k}+1}{}^{q}\right)^{w_{\text{k}+1}}}{\left(1+(\gamma-1)n_{\text{k}+1}{}^{q}\right)^{w_{\text{k}+1}}+(\gamma-1)\left(1-n_{\text{k}+1}{}^{q}\right)^{w_{\text{k}+1}}}} e^{i2\pi\sqrt[q]{\frac{\left(1+(\gamma-1)\varpi_{n_{\text{k}+1}}{}^{q}\right)^{w_{\text{k}+1}}-\left(1-\varpi_{n_{\text{k}+1}}{}^{q}\right)^{w_{\text{k}+1}}}{\left(1+(\gamma-1)\varpi_{n_{\text{k}+1}}{}^{q}\right)^{w_{\text{k}+1}}+(\gamma-1)\left(1-\varpi_{n_{\text{k}+1}}{}^{q}\right)^{w_{\text{k}+1}}}}}\end{array}\right)$$

C*p,q*-$\mathcal{R}$OFHWG$(\text{Ŧ}_1, \text{Ŧ}_2, \text{Ŧ}_3 \ldots \text{Ŧ}_{\text{k}+1}) =$

$$\left(\begin{array}{c}\frac{\sqrt[p]{\gamma}\prod_{\text{j}=1}^{\text{k}+1} m_{\text{j}}^{w_{\text{j}}}}{\sqrt[p]{\prod_{\text{j}=1}^{\text{k}+1}\left(1+(\gamma-1)\left(1-m_{\text{j}}{}^{p}\right)\right)^{w_{\text{j}}}+(\gamma-1)\prod_{\text{j}=1}^{\text{k}+1}\left(m_{\text{j}}{}^{p}\right)^{2w_{\text{j}}}}} e^{i2\pi\frac{\sqrt[p]{\gamma}\prod_{\text{j}=1}^{\text{k}+1}\varpi_{m_{\text{j}}}{}^{w_{\text{j}}}}{\sqrt[p]{\prod_{\text{j}=1}^{\text{k}+1}\left(1+(\gamma-1)\left(1-\varpi_{m_{\text{j}}}{}^{p}\right)\right)^{w_{\text{j}}}+(\gamma-1)\prod_{\text{j}=1}^{\text{k}+1}\left(m^{p}\right)^{2w_{\text{j}}}}}} \\ \sqrt[q]{\frac{\prod_{\text{j}=1}^{\text{k}+1}\left(1+(\gamma-1)n_{\text{j}}{}^{q}\right)^{w_{\text{j}}}-\prod_{\text{j}=1}^{\text{k}+1}\left(1-n_{\text{j}}{}^{q}\right)^{w_{\text{j}}}}{\prod_{\text{j}=1}^{\text{k}+1}\left(1+(\gamma-1)n_{\text{j}}{}^{q}\right)^{w_{\text{j}}}+(\gamma-1)\prod_{\text{j}=1}^{\text{k}+1}\left(1-n_{\text{j}}{}^{q}\right)^{w_{\text{j}}}}} e^{i2\pi\sqrt[q]{\frac{\prod_{\text{j}=1}^{\text{k}+1}\left(1+(\gamma-1)\varpi_{n_{\text{j}}}{}^{q}\right)^{w_{\text{j}}}-\prod_{\text{j}=1}^{\text{k}+1}\left(1-\varpi_{n_{\text{j}}}{}^{q}\right)^{w_{\text{j}}}}{\prod_{\text{j}=1}^{\text{k}+1}\left(1+(\gamma-1)\varpi_{n_{\text{j}}}{}^{q}\right)^{w_{\text{j}}}+(\gamma-1)\prod_{\text{j}=1}^{\text{k}+1}\left(1-\varpi_{n_{\text{j}}}{}^{q}\right)^{w_{\text{j}}}}}}\end{array}\right)$$

The result is valid for all values of l and for $\text{l} = \text{k} + 1$.

In this theorem, the fundamental characteristics of the proposed C*p,q*-$\mathcal{R}$OFHWG operator are stated.

Theorem 6: Following are the properties of the Hamacher geometric aggregation operator of C*p,q*-$\mathcal{R}$OFNs:

i. (Idempotency) If $\text{Ŧ}_j = \text{Ŧ} = \left(m_i e^{i2\pi\varpi_{m_i}}, n_i e^{i2\pi\varpi_{n_i}}\right) = \left(m e^{i2\pi\varpi_m}, n e^{i2\pi\varpi_n}\right), \forall\, j = 1,2,3 \ldots \text{l}$.
C*p,q*-$\mathcal{R}$OFHWG $(\text{Ŧ}_1, \text{Ŧ}_2, \text{Ŧ}_3 \ldots \text{Ŧ}_n) = \text{Ŧ}$.

ii. (Boundedness) If $\text{Ŧ}^- = \left(\min_{\text{j}} m_{\text{j}}, \max_{\text{j}} n_{\text{j}}\right)$ and $\text{Ŧ}^+ = \left(\min_{\text{j}} m_{\text{j}}, \max_{\text{j}} n_{\text{j}}\right)$. Then
$\text{Ŧ}^- \leq$ C*p,q*-$\mathcal{R}$OFHWG $(\text{Ŧ}_1, \text{Ŧ}_2, \text{Ŧ}_3 \ldots \text{Ŧ}_n) \leq \text{Ŧ}^+$.

iii. (Monotonically) Let Ŧ_{j} and P_{j} be two C*p,q*-$\mathcal{R}$OFNs such that $\text{Ŧ}_{\text{j}} \leq P_{\text{j}} \forall \text{j}$.Then C*p,q*-$\mathcal{R}$OFHWG $(\text{Ŧ}_1, \text{Ŧ}_2, \text{Ŧ}_3 \ldots \text{Ŧ}_n) \leq$ C*p,q*-$\mathcal{R}$OFHWG $(P_1, P_2, P_3 \ldots P_n)$.

This is demonstrable in an analogous way. Weighting the C*p,q*-$\mathcal{R}$OFN alone is done by the C*p,q*-$\mathcal{R}$OFHWG aggregation operator. There are situations in which the C*p,q*-$\mathcal{R}$OFN's ordered position is important in MADM problems. In those cases, the idea of ordered weighted averaging operators is important, and the C*p,q*-$\mathcal{R}$OFHOWGO is suggested as a solution.

Definition 14: Let $Ŧ = \left(m_i e^{i2\pi\varpi_{m_i}}, n_i e^{i2\pi\varpi_{n_i}}\right)$ be a collection. The form of C*p,q*-$\mathcal{R}$OFHOWGO is mapping $T^n \rightharpoonup T$

C*p,q*-$\mathcal{R}$OFHOWG $(Ŧ_1, Ŧ_2, Ŧ_3 \ldots Ŧ_n) = \bigoplus_{ȷ=1}^{l} w_ȷ Ŧ_{\sigma(ȷ)}$ where $\sigma(ȷ)$ is such that $Ŧ_{\sigma(ȷ-1)} \geq Ŧ_{\sigma(ȷ)} \forall j$.

Theorem 7: Let $Ŧ = \left(m_i e^{i2\pi\varpi_{m_i}}, n_i e^{i2\pi\varpi_{n_i}}\right)$ = is a collection. Then C*p,q*-$\mathcal{R}$OFHOWGO with form C*p,q*-$\mathcal{R}$OFHOWG $(Ŧ_1, Ŧ_2, Ŧ_3 \ldots Ŧ_n) = \bigoplus_{ȷ=1}^{l} w_ȷ Ŧ_{\sigma(ȷ)}$

$$\left(\begin{array}{c} \frac{\sqrt[p]{\gamma}\prod_{ȷ=1}^{l} m_{\sigma(ȷ)}^{w_ȷ}}{\sqrt[p]{\prod_{ȷ=1}^{l}\left(1+(\gamma-1)\left(1-m_{\sigma(ȷ)}{}^{p}\right)\right)^{w_ȷ}+(\gamma-1)\prod_{ȷ=1}^{l}\left(m_{\sigma(ȷ)}{}^{p}\right)^{2w_ȷ}}} e^{i2\pi\frac{\sqrt[p]{\gamma}\prod_{ȷ=1}^{l}\varpi_{m_{\sigma(ȷ)}}^{w_ȷ}}{\sqrt[p]{\prod_{ȷ=1}^{l}\left(1+(\gamma-1)\left(1-\varpi_{m_{\sigma(ȷ)}}{}^{p}\right)\right)^{w_ȷ}+(\gamma-1)\prod_{ȷ=1}^{l}\left(\varpi_{m_{\sigma(ȷ)}}{}^{p}\right)^{2w_ȷ}}}} \\ \sqrt[q]{\frac{\prod_{ȷ=1}^{l}\left(1+(\gamma-1)n_{\sigma(ȷ)}{}^{q}\right)^{w_ȷ}-\prod_{ȷ=1}^{l}\left(1-n_{\sigma(ȷ)}{}^{q}\right)^{w_ȷ}}{\prod_{ȷ=1}^{l}\left(1+(\gamma-1)n_{\sigma(ȷ)}{}^{q}\right)^{w_ȷ}+(\gamma-1)\prod_{ȷ=1}^{l}\left(1-n_{\sigma(ȷ)}{}^{q}\right)^{w_ȷ}}} e^{i2\pi\sqrt[q]{\frac{\prod_{ȷ=1}^{l}\left(1+(\gamma-1)\varpi_{n_{\sigma(ȷ)}}{}^{q}\right)^{w_ȷ}-\prod_{ȷ=1}^{l}\left(1-\varpi_{n_{\sigma(ȷ)}}{}^{q}\right)^{w_ȷ}}{\prod_{ȷ=1}^{l}\left(1+(\gamma-1)\varpi_{n_{\sigma(ȷ)}}{}^{q}\right)^{w_ȷ}+(\gamma-1)\prod_{ȷ=1}^{l}\left(1-\varpi_{n_{\sigma(ȷ)}}{}^{q}\right)^{w_ȷ}}}} \end{array}\right) \tag{11}$$

The C*p,q*-$\mathcal{R}$OFHWG operator weighs ordered locations in Eqs. (10) and (11), whereas the C*p,q*-$\mathcal{R}$OFHOWG operator weighs arguments directly. We suggest a hybrid geometric operator as a solution.

Definition 15: [19] Let $Ŧ = \left(m_i e^{i2\pi\varpi_{m_i}}, n_i e^{i2\pi\varpi_{n_i}}\right)$ be a collection. C*p,q*-$\mathcal{R}$OFHHG operator mapping $T^n \rightharpoonup T$;

C*p,q*-$\mathcal{R}$OFHHG $(Ŧ_1, Ŧ_2, Ŧ_3 \ldots Ŧ_n) = \bigoplus_{ȷ=1}^{l} w_ȷ \dot{Ŧ}_{\sigma(ȷ)}$

$\dot{Ŧ}_{\sigma(ȷ)}$ is jth largest of the $\mathcal{R}$OFN $\dot{Ŧ}_ȷ = Ŧ_ȷ^{l\omega_ȷ}$ with $\omega_ȷ$ is weight vector of C*p,q*-$\mathcal{R}$OF argument $Ŧ_ȷ$ and $\omega_ȷ \in [0,1]$ and $\sum_1^n \omega_ȷ$ = 1and l is a balancing coefficient.

Theorem 8: Let $Ŧ = \left(m_i e^{i2\pi\varpi_{m_i}}, n_i e^{i2\pi\varpi_{n_i}}\right)$ is a collection. The form of C*p,q*-$\mathcal{R}$OFHHGO is TSFHHG$(Ŧ_1, Ŧ_2, Ŧ_3 \ldots Ŧ_n)$

$$\left(\begin{array}{c}\sqrt[p]{\frac{\prod_{j=1}^{l}\left(1+(\gamma-1)\dot{n}_{\sigma(j)}{}^{p}\right)^{w_j}-\prod_{j=1}^{l}\left(1-\dot{n}_{\sigma(j)}{}^{p}\right)^{w_j}}{\prod_{j=1}^{l}\left(1+(\gamma-1)\dot{n}_{\sigma(j)}{}^{p}\right)^{w_j}+(\gamma-1)\prod_{j=1}^{l}\left(1-\dot{n}_{\sigma(j)}{}^{p}\right)^{w_j}}}\,e^{i2\pi\sqrt[p]{\frac{\prod_{j=1}^{l}\left(1+(\gamma-1)\varpi_{\dot{n}_{\sigma(j)}}{}^{p}\right)^{w_j}-\prod_{j=1}^{l}\left(1-\varpi_{\dot{n}_{\sigma(j)}}{}^{p}\right)^{w_j}}{\prod_{j=1}^{l}\left(1+(\gamma-1)\varpi_{\dot{n}_{\sigma(j)}}{}^{p}\right)^{w_j}+(\gamma-1)\prod_{j=1}^{l}\left(1-\varpi_{\dot{n}_{\sigma(j)}}{}^{q}\right)^{w_j}}}} \\ \frac{\sqrt[q]{\gamma}\prod_{j=1}^{l}\dot{m}_{\sigma(j)}^{w_j}}{\sqrt[q]{\prod_{j=1}^{l}\left(1+(\gamma-1)\left(1-\dot{m}_{\sigma(j)}{}^{q}\right)\right)^{w_j}+(\gamma-1)\prod_{j=1}^{l}\left(\dot{m}_{\sigma(j)}{}^{q}\right)^{2w_j}}}\,e^{i2\pi\frac{\sqrt[q]{\gamma}\prod_{j=1}^{l}\varpi_{\dot{m}_{\sigma(j)}}^{w_j}}{\sqrt[q]{\prod_{j=1}^{l}\left(1+(\gamma-1)\left(1-\varpi_{\dot{m}_{\sigma(j)}}{}^{q}\right)\right)^{w_j}+(\gamma-1)\prod_{j=1}^{l}\left(\varpi_{\dot{m}_{\sigma(j)}}{}^{q}\right)^{2w_j}}}}\end{array}\right) \quad (12)$$

3.4 Algorithm

This section proposes an application to tackle MAGDM problems based on sophisticated *p,q*-ROFHO. We have two finite sets of alternative and attributes $G=\{G_1,G_2,G_3,\ldots,G_m\}$ and $A=\{A_1,A_2,A_3,\ldots,A_n\}$ and decision makers $D=\{D_1,D_2,D_3,\ldots,D_k\}$. The weight vector $w=\{w_1,w_2,w_3,\ldots w_n\}^T$ is given for attributes $A_i(i=1,2,3,\ldots n)$ with a condition $\sum_{i=1}^{n}w_i=1$, $w_i\epsilon[0,1]$. The following steps make up the method for aggregating the C*p,q*-$\mathcal{R}$OF information.

Step 1: Use Eq. (13) to construct the matrix, where each entity is represented by a C*p,q*-$\mathcal{R}$OFN.

$$D^s=\left[Ŧ_{ij}^{S}\right]_{m\times n},\, i,j=1,2,\ldots m,n \tag{13}$$

Step 2: Normalization of decision matrix by using Eq.14.

$$R^s=\begin{cases}Ŧ_{ij}^{S}\ for\ benefite\ type\ of\ criteria\\ Ŧ_{ij}^{S\,C}\ for\ cost\ type\ of\ criteria\end{cases} \tag{14}$$

Step 3: Examine the optimal levels of each criterion while building the optimal approach using Eq. (14).

$$R=\begin{bmatrix}r_{11}\\r_{12}\\r_{13}\\\cdots\\r_{1n}\end{bmatrix} \tag{15}$$

Step 4: Using Eq. (7–12), aggregate the decision matrix with optimal approach.

Step 5: Gather the values of the options in ascending order, then select the one that the decision-makers believe is best.

3.4.1 Criteria for assessing gold miners' cleaner production

The cleaner production evaluation criteria system, which employs five criteria based on the distinctive features of gold mines, was put into practice in this section using the recommended ranking technique. To determine which of the three solutions is the best for cleaner manufacturing. Decision maker [20] characterizes cleaner manufacturing based on five criteria, mentioned in Table 1. Figure 1 represents the flow chart of Algorithm. The algorithm's steps are:

Step 1: By using Eq. (13) to construct a matrix, where each entity is represented by a C*p,q*-$\mathcal{R}$OFNs.

Step 2: Use Eq. (14) for normalization of decision matrix.

Step 3: Use matrix (15) to build the optimal approach, looking at the optimal levels of each criterion.

Step 4: The best method was used to aggregate by Eqs.7–12.

Step 5: Sort the choice values according to their order of ascending, then choose the best option based on the preferences of the decision-makers.

The following are the steps in the algorithm in Figure 2:

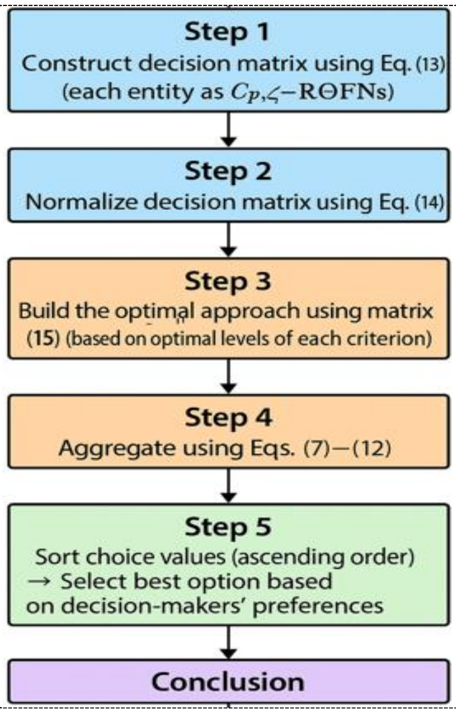


Fig. 2. **Algorithmic Framework for Complex p,q-Rung Orthopair Fuzzy Hamacher Aggregation-Based MAGDM**

Tables 4–26, which are presented below, include numerical discussions of the information of CIFSs, CPFSs and C*p,q*-$\mathcal{R}$OFSs together with their associated results.

Step 1: We use Eq. (13), whose every entity is a C*p,q*-$\mathcal{R}$OFN, to generate the matrix. The following is the decision matrix (Table 2):

Step 2: It is not possible to normalize the choice matrix using Eq. (14). Therefore, Table 1 will be taken into consideration for computations using weight vectors.

$$w = \{w_1, w_2, w_3, \dots w_n\}^T.$$

Step 3: Using Eq. (15), we create the optimal plan and analyze the optimal values of each criterion so that

$$
\text{R} = \begin{bmatrix} \text{Level of management} & \left(1.0e^{\iota 2\pi(1.0)} 0.0e^{\iota 2\pi(0.0)}\right) \\ \text{Equipment and production method} & \left(1.0e^{\iota 2\pi(1.0)} 0.0e^{\iota 2\pi(0.0)}\right) \\ \text{Use of resources and energy} & \left(1.0e^{\iota 2\pi(1.0)} 0.0e^{\iota 2\pi(0.0)}\right) \\ \text{Utilization of waste} & \left(1.0e^{\iota 2\pi(1.0)} 0.0e^{\iota 2\pi(0.0)}\right) \\ \text{The ecological conditions} & \left(1.0e^{\iota 2\pi(1.0)} 0.0e^{\iota 2\pi(0.0)}\right) \end{bmatrix}
$$

Step 4: Using Eq. (7), the Cp,q-$\mathcal{R}$OF decision matrix with optimal method was aggregated (Table 3).

Step 5: Compile the alternative values in ascending order, then select the option that the decision makers believe is best (Tables 4, 5).

3.4.2 Practical Case Illustration and Justification

To evaluate the practical utility of the proposed Cp,q-$\mathcal{R}$OFHA operator, we consider a real-world-inspired case involving the assessment of Cleaner Production (CP) strategies in a mid-scale gold mining operation. The goal is to select the most sustainable alternative from a set of proposed mining practices based on environmental, economic, and operational criteria.

The decision-making committee consists of domain experts, environmental engineers, and sustainability officers who evaluate five CP alternatives**:**

A_1: Enhanced tailings management

A_2: Closed-loop water recycling systems

A_3: Low-toxicity chemical substitution

A_4: Renewable-energy-powered machinery

A_5: Waste rock repurposing techniques

Each alternative is assessed across five key attributes relevant to sustainability in gold mining:

1. Reduction in Environmental Impact
2. Cost-Efficiency
3. Technical Feasibility
4. Resource Optimization
5. Regulatory Compliance

Expert evaluations are expressed using Complex p, q-Rung Orthopair Fuzzy Numbers, allowing them to capture both hesitation and partial agreement (via complex-valued judgments). These evaluations are then aggregated using the proposed Hamacher operators. In the real-world setting of gold mining, CP adoption is often hindered by uncertainty in expert judgment, especially when sustainability trade-offs involve conflicting priorities (e.g., cost vs. environmental performance). Our model offers a robust and flexible tool for mining companies and environmental regulators to synthesize multiple expert opinions, even when the data is imprecise or conflicting. The ability to process such uncertainty with complex fuzzy logic leads to more stable and well-grounded decisions**.** By using the Cp,q-$\mathcal{R}$OFS framework:

1. Experts can express more nuanced judgments**.**
2. The aggregation process preserves information richness (via complex degrees).
3. The final rankings are resilient to changes in criteria weights (as shown in the sensitivity analysis).

The comparative analysis, as shown in Table 5 and Figure 3, demonstrates that the proposed model results in a more discriminative and stable ranking of alternatives, making it suitable for sustainability-driven industries like mining.

Example 2: The details pertaining to this instance were covered previously. The algorithm's steps are:

Step 1: The matrix is created using Eq. (13), where each entity is represented by a CPFN. The choice matrix is shown in Table 6 as follows:

Step 2: Eq. (14) cannot be used to normalize the decision matrix. For computations involving weight vectors $w = \{0.3,0.3,0.4\}^T$.

Step 3: We use Eq. (15) to create the optimal strategy and analyze the optimal levels of each criterion so that

$$R = \begin{bmatrix} \text{Level of management} & \left(1.0e^{\iota 2\pi(1.0)} 0.0e^{\iota 2\pi(0.0)}\right) \\ \text{Equipment and production method} & \left(1.0e^{\iota 2\pi(1.0)} 0.0e^{\iota 2\pi(0.0)}\right) \\ \text{Use of resources and energy} & \left(1.0e^{\iota 2\pi(1.0)} 0.0e^{\iota 2\pi(0.0)}\right) \\ \text{Utilization of waste} & \left(1.0e^{\iota 2\pi(1.0)} 0.0e^{\iota 2\pi(0.0)}\right) \\ \text{The ecological conditions} & \left(1.0e^{\iota 2\pi(1.0)} 0.0e^{\iota 2\pi(0.0)}\right) \end{bmatrix}$$

Step 4: Eq. (7) was used to aggregate with the optimal approach (Table 7).

Step 5: Compile the alternative values in ascending order, then select the option that the decision makers believe is preferable (Table 8, 9).

Step 6: Conclusion.

Example 3: The details pertaining to this instance were covered previously. The steps of algorithm are:

Step 1: Using Eq. (13), whose entities are all CIFNs, we build the matrix. The following is the decision matrix (Table 10):

Step 2: It is not possible to use Eq. (14) to normalize the decision matrix. As a result, we shall take weight vector $w = \{0.3,0.3,0.4\}^T$ computations.

Step 3: Using Eq. (15), we create the optimal plan and analyze the optimal values of each criterion so that

$$\text{R} = \begin{bmatrix} \text{Level of management} & \left(1.0e^{\iota 2\pi(1.0)}0.0e^{\iota 2\pi(0.0)}\right) \\ \text{Equipment and production method} & \left(1.0e^{\iota 2\pi(1.0)}0.0e^{\iota 2\pi(0.0)}\right) \\ \text{Use of resources and energy} & \left(1.0e^{\iota 2\pi(1.0)}0.0e^{\iota 2\pi(0.0)}\right) \\ \text{Utilization of waste} & \left(1.0e^{\iota 2\pi(1.0)}0.0e^{\iota 2\pi(0.0)}\right) \\ \text{The ecological conditions} & \left(1.0e^{\iota 2\pi(1.0)}0.0e^{\iota 2\pi(0.0)}\right) \end{bmatrix}$$

Step 4: Using Eq. (7) (Table 11), the C*p,q*-$\mathcal{R}$OF decision matrix was aggregated with optimal approach.

Step 5: Compile the alternative values in ascending order, then select the option that the decision makers believe is best (Tables 12, 13).

Example 4: This example's specifics were already discussed.

Step 1: Use Eq. (13), whose every entity is a *p,q*-$\mathcal{R}$OFN, to generate the matrix. The following is the matrix of the decision with Tables 14, 15. Since $e^0 = 1$, then a matrix with each entity represented by a complex number.

Step 2: Eq. (14) cannot be used to normalize the decision matrix. For weight vector calculations $w = \{0.3,0.3,0.4\}^T$.

Step 3: Equation (15) is used to design the optimal approach, and the ideal levels of each criterion are examined so that

$$\text{R} = \begin{bmatrix} \text{Level of management} & \left(1.0e^{\iota 2\pi(1.0)}0.0e^{\iota 2\pi(0.0)}\right) \\ \text{Equipment and production method} & \left(1.0e^{\iota 2\pi(1.0)}0.0e^{\iota 2\pi(0.0)}\right) \\ \text{Use of resources and energy} & \left(1.0e^{\iota 2\pi(1.0)}0.0e^{\iota 2\pi(0.0)}\right) \\ \text{Utilization of waste} & \left(1.0e^{\iota 2\pi(1.0)}0.0e^{\iota 2\pi(0.0)}\right) \\ \text{The ecological conditions} & \left(1.0e^{\iota 2\pi(1.0)}0.0e^{\iota 2\pi(0.0)}\right) \end{bmatrix}$$

Step 4: Using Eq. (7) (Tables 16, 17, 18), the C*p,q*-$\mathcal{R}$OF decision matrix was aggregated with optimal approach.

Step 5: Sort the values of the options in ascending order, then select the one that the decision-makers believe is best.

Example 5: The details pertaining to this instance were covered previously. The algorithm's as:

Step 1: From Eq. (13), A constructed matrix with all entities have form of PFNs and by following table 19.

Step 2: Eq. (14) cannot be used to normalize the decision matrix. For computations involving weight vectors $w = \{0.3,0.3,0.4\}^T$.

Step 3: Using Eq. (15), we create the optimal plan and analyze the optimal values of each criterion so that

$$R = \begin{bmatrix} \text{Level of management} & \left(1.0e^{\iota 2\pi(1.0)} 0.0e^{\iota 2\pi(0.0)}\right) \\ \text{Equipment and production method} & \left(1.0e^{\iota 2\pi(1.0)} 0.0e^{\iota 2\pi(0.0)}\right) \\ \text{Use of resources and energy} & \left(1.0e^{\iota 2\pi(1.0)} 0.0e^{\iota 2\pi(0.0)}\right) \\ \text{Utilization of waste} & \left(1.0e^{\iota 2\pi(1.0)} 0.0e^{\iota 2\pi(0.0)}\right) \\ \text{The ecological conditions} & \left(1.0e^{\iota 2\pi(1.0)} 0.0e^{\iota 2\pi(0.0)}\right) \end{bmatrix}$$

Step 4: By using Eq. (7), with optimal method to aggregate and by 20, 21, 22, 23 tables.

Step 5: Compile the alternative values in ascending order, then select best.

Example 6: All details of this instance were already covered. The algorithm's steps are as follows:

Step I: Using Eq. (13), whose entities are all complex IFN, we build the matrix. The following is the decision matrix:

Step 2: It is not possible to normalize the decision matrix using Equation (14). Therefore, computations involving weight vectors $w = \{0.3,0.3,0.4\}^T$.

Step 3: We use Eq. (15) to create the optimal strategy and analyze the optimal levels of each criterion so that

$$R = \begin{bmatrix} \text{Level of management} & \left(1.0e^{\iota 2\pi(1.0)} 0.0e^{\iota 2\pi(0.0)}\right) \\ \text{Equipment and production method} & \left(1.0e^{\iota 2\pi(1.0)} 0.0e^{\iota 2\pi(0.0)}\right) \\ \text{Use of resources and energy} & \left(1.0e^{\iota 2\pi(1.0)} 0.0e^{\iota 2\pi(0.0)}\right) \\ \text{Utilization of waste} & \left(1.0e^{\iota 2\pi(1.0)} 0.0e^{\iota 2\pi(0.0)}\right) \\ \text{The ecological conditions} & \left(1.0e^{\iota 2\pi(1.0)} 0.0e^{\iota 2\pi(0.0)}\right) \end{bmatrix}$$

Step 4: Using the best method, the C*p,q*-$\mathcal{R}$OF decision matrix was aggregated using Eq. (7) (Tables 24, 25, 26).

Step 5: Choose the choice that the decision-makers think is best after sorting the values of the options in ascending order.

3.4.3 Comparative Evaluation

The proposed Complex p,q-Rung Orthopair Fuzzy Sets (Cp,q-$\mathcal{R}$OFSs) offer a flexible and comprehensive structure for modeling complex fuzzy (CF) information often encountered in real-world decision-making problems. Compared to conventional models such as Complex Intuitionistic Fuzzy Sets (CIFS) and Complex Pythagorean Fuzzy Sets (CPFS), Cp,q-$\mathcal{R}$OFSs allow dynamic control of information bounds through tunable parameters p and q, thus increasing their expressiveness. The Hamacher aggregation operators (HAOs) serve as key instruments in capturing the interrelationships among uncertain inputs and constructing robust operational rules. A comparative analysis has been conducted to demonstrate the superiority of our proposed model. To evaluate performance, we solved the same multi-attribute group decision-making (MAGDM) problem using several existing methods: Xu (2007) – Intuitionistic Fuzzy Aggregation Operator (IFAO), Huang (2014) – Intuitionistic Fuzzy Hamacher Aggregation Operator (IFHAO), Garg (2019) – Pythagorean Fuzzy Hamacher Aggregation Operator (PFHAO), Liu and Wang (2018a) – p,q -ROF Aggregation Operator. In contrast, our proposed method—based on Cp,q-$\mathcal{R}$OFHA and Cp,q-$\mathcal{R}$OFGH operators—provides the following comparative advantages: Higher precision in representing expert opinions through complex-valued membership and non-membership degrees. Enhanced robustness to parameter variations, as observed in sensitivity analysis with respect to p,q and γ. Generalization ability, where our model reduces to existing ones (e.g., CIFS, IF, PF, and p,q -ROFS) under specific parameter constraints. The results of this comparative evaluation are summarized as: Table 1: Evaluation of CP alternatives, Table 2: Normalized decision matrix, Table 3: Aggregated decision values using different models, Table 4: Calculated score values, Table 5: Final ranking of alternatives, Figure 3: Comparative performance of criteria under various aggregation methods. Overall, the findings confirm that the Cp,q-$\mathcal{R}$OFS-based Hamacher aggregation framework outperforms previous methods in terms of ranking consistency, decision accuracy, and flexibility in managing uncertain and complex expert assessments.

Table 1

Description of assessment of CP alternatives

notations	depiction	Descriptive
C_1	Stages of management	It outlines the cleaner production management level, including the essential elements of cleaner production rules and how they should be implemented.
C_2	Equipment and production method	It displays the scope of the production process, including industrial equipment and extraction methods.
C_3	Use of resources and energy	It details how much water is used for each unit of product and how much energy is needed overall for each unit of production.
C_4	Waste optimization	It details overall use, including rates of use of solid waste, wastewater, and related resources.
C_5	The ecological conditions	Criteria weights, greening and reclamation rates, and ecological governance are all included.

Table 2

Normalized decision matrix

Symbols	G_1	G_2	G_3
C_1	$(0.77e^{\iota 2\pi(0.52)}, 0.41e^{\iota 2\pi(0.47)})$	$(0.87e^{\iota 2\pi(.52)}, 0.71e^{\iota 2\pi(0.47)})$	$(0.92e^{\iota 2\pi(0.58)}, 0.88e^{\iota 2\pi(0.47)})$
C_2	$(0.85e^{\iota 2\pi(0.55)}, 0.74e^{\iota 2\pi(0.45)})$	$(0.85e^{\iota 2\pi(0.55)}, 0.74e^{\iota 2\pi(0.35)})$	$(0.92e^{\iota 2\pi(0.56)}, 0.86e^{\iota 2\pi(0.55)})$
C_3	$(0.9e^{\iota 2\pi(0.58)}, 0.75e^{\iota 2\pi(0.51)})$	$(0.83e^{\iota 2\pi(0.51)}, 0.75e^{\iota 2\pi(0.41)})$	$(0.9e^{\iota 2\pi(0.55)}, 0.85e^{\iota 2\pi(0.41)})$
C_4	$(0.7e^{\iota 2\pi(0.6)} 0.5e^{\iota 2\pi(0.54)})$	$(0.89e^{\iota 2\pi(0.6)} 0.72e^{\iota 2\pi(0.21)})$	$(0.79e^{\iota 2\pi(0.59)} 0.89e^{\iota 2\pi(0.29)})$
C_5	$(0.84e^{\iota 2\pi(0.45)}, 0.83e^{\iota 2\pi(0.44)})$	$(0.86e^{\iota 2\pi(0.45)}, 0.74e^{\iota 2\pi(0.25)})$	$(0.94e^{\iota 2\pi(0.57)}, 0.87e^{\iota 2\pi(0.45)})$

Table 3

C*p*,*q*-$\mathcal{R}$OFNs Aggregated values

Method	Values of alternatives
C_1	$(0.87e^{\iota 2\pi(0.54)}, 0.56e^{\iota 2\pi(0.39)})$
C_2	$(0.88e^{\iota 2\pi(0.55)}, 0.68e^{\iota 2\pi(0.38)})$
C_3	$(0.88e^{\iota 2\pi(0.54)}, 0.68e^{\iota 2\pi(0.37)})$
C_4	$(0.80e^{\iota 2\pi(0.59)} 0.60e^{\iota 2\pi(0.26)})$
C_5	$(0.89e^{\iota 2\pi(0.50)}, 0.72e^{\iota 2\pi(0.31)})$

Table 4

C*p*,*q*-$\mathcal{R}$OFNs Score values

Score values
$\dot{\check{S}}(C_1) = \mathbf{0.67}$
$\dot{\check{S}}(C_2) = \mathbf{0.65}$
$\dot{\check{S}}(C_3) = \mathbf{0.65}$
$\dot{\check{S}}(C_4) = \mathbf{0.64}$
$\dot{\check{S}}(C_5) = \mathbf{0.63}$

Table 5

Ranking values for C*p*,*q*-$\mathcal{R}$OFNs

Ranking values	Optimal choice
$C_1 \geq C_2 \geq C_3 \geq C_4 \geq C_5$	C_1

Table 6 represents the normalized decision matrix. Table 7 shows the aggregated value. Table 8 shows the scores of alternatives and Table 9 represents the ranking of alternatives.

Table 6

Normalized decision matrix

Symbols	G_1	G_2	G_3
C_1	$(0.9e^{\iota 2\pi(0.58)}, 0.2e^{\iota 2\pi(0.51)})$	$(0.83e^{\iota 2\pi(.51)}, 0.28e^{\iota 2\pi(0.41)})$	$(0.9e^{\iota 2\pi(0.55)}, 0.11e^{\iota 2\pi(0.41)})$
C_2	$(0.85e^{\iota 2\pi(0.55)}, 0.3e^{\iota 2\pi(0.45)})$	$(0.85e^{\iota 2\pi(0.55)}, 0.22e^{\iota 2\pi(0.35)})$	$(0.92e^{\iota 2\pi(0.56)}, 0.1e^{\iota 2\pi(0.55)})$
C_3	$(0.84e^{\iota 2\pi(0.45)}, 0.3e^{\iota 2\pi(0.44)})$	$(0.86e^{\iota 2\pi(0.45)}, 0.23e^{\iota 2\pi(0.25)})$	$(0.94e^{\iota 2\pi(0.57)}, 0.08e^{\iota 2\pi(0.45)})$
C_4	$(0.77e^{\iota 2\pi(0.52)} 0.33e^{\iota 2\pi(0.47)})$	$(0.87e^{\iota 2\pi(0.52)} 0.24e^{\iota 2\pi(0.47)})$	$(0.92e^{\iota 2\pi(0.58)} 0.09e^{\iota 2\pi(0.47)})$
C_5	$(0.7e^{\iota 2\pi(0.6)}, 0.4e^{\iota 2\pi(0.54)})$	$(0.89e^{\iota 2\pi(0.6)}, 0.2e^{\iota 2\pi(0.21)})$	$(0.79e^{\iota 2\pi(0.59)}, 0.23e^{\iota 2\pi(0.29)})$

Table 7

Values after aggregation

Method	Values of alternatives
C_1	$(0.88e^{\iota 2\pi(0.54)}, 0.14e^{\iota 2\pi(0.37)})$
C_2	$(0.87e^{\iota 2\pi(0.55)}, 0.14e^{\iota 2\pi(0.38)})$
C_3	$(0.89e^{\iota 2\pi(0.50)}, 0.13e^{\iota 2\pi(0.31)})$
C_4	$(0.87e^{\iota 2\pi(0.54)}, 0.15e^{\iota 2\pi(0.39)})$
C_5	$(0.80e^{\iota 2\pi(0.59)}, 0.69e^{\iota 2\pi(0.26)})$

Table 8

Values of Score

Score values
$\dot{\check{S}}(C_1) = \mathbf{0.70}$
$\dot{\check{S}}(C_2) = \mathbf{0.70}$
$\dot{\check{S}}(C_3) = \mathbf{0.70}$
$\dot{\check{S}}(C_4) = \mathbf{0.69}$
$\dot{\check{S}}(C_5) = \mathbf{0.62}$

Table 9

Ranking values

Ranking values	Optimal choice
$C_1 \geq C_2 \geq C_3 \geq C_4 \geq C_5$	$C_1 \rightarrow 1.0$

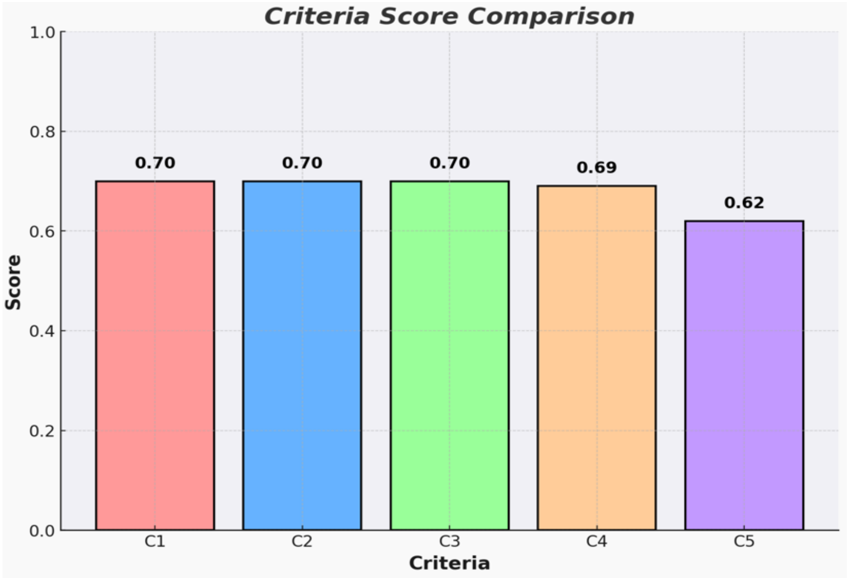


Figure 3: Comparison of Criteria

3.4.4 Sensitive analysis

The following is a list of the main benefits of the examined approaches:

i. The operators of the Cp,q-$\mathcal{R}$OF Hamacher aggregation generalize existing fuzzy Hamacher aggregation operations.

ii. We compare Cp,q-$\mathcal{R}$OFHAOs to the complex Pythagorean and complex intuitionistic fuzzy HAOs, the latter of which additionally takes the refusal grade into account.

iii. Cp,q-$\mathcal{R}$OFHAOs can give the solution of issues examined, but the present HAOs are unable to manage the difficulties outlined in the Cp,q-$\mathcal{R}$OF environment.

iv. Cp,q-$\mathcal{R}$OFHAOs yield more stable results than those seen in the study.

The Cp,q-$\mathcal{R}$OFSs basically consists of two functions: complex valued membership and non-membership grades. The suggested strategy has benefits, that if,

$\text{Ą} = (\acute{m}_E, \acute{n}_E) = \left(m_E(\tilde{v}).\, e^{\hat{\imath}\, 2\pi\varphi_{m_E(\tilde{v})}}, n_E(\hat{\tilde{v}}).\, e^{\hat{\imath}\, 2\pi\varphi_{n_E(\tilde{v})}}\right)$ represents the Cp,q-$\mathcal{R}$OFN Consequently, the following circumstances are met:

$0 \leq m_E^p + n_E^q \leq 1, 0 \leq \varphi_{m_E}^p + \varphi_{m_E}^q \leq 1, p, q \geq 1$. From the limitations of Cp,q-$\mathcal{R}$OFS, it is evident that the specific instances of the suggested method are the CIFSs and CPFSs. Considering everything discussed above, it is evident that the procedures suggested in this article are more accurate and dependable than those now in use. Decision makers can reflect their pessimistic or optimistic attitude by suitably choosing among the many Hamacher weighted aggregation operators suggested in this work, all while keeping in mind the actual demands. Additionally, our approach differs from current ones in that it incorporates extra criteria that represent the preferences of decision-makers, enabling them to make decisions using risk tolerance. Compared to Wu and Wei (2017), Garg (2017), Liu and

Wang (2018a), and Darko and Liang (2020), our approach is more realistic, effective, and logical. Huang (2014) used IFHA operators, Xu (2007) used IFA operators, and Garg (2019) expanded on IFHA operators as specific instances of the suggested methods. The linguistic neutrosophic set, interval type-2 fuzzy set, probabilistic linguistic information, linguistic D Number, and T-spherical fuzzy set are only a few of the fuzzy sets for which we will investigate Hamacher aggregation strategies in the future. Table 10 illustrates normalized decision matrix. Table 11 shows the aggregated values by using the complex p,q rung orthopair fuzzy sets. Table 12 represents the score value and ranking of alternatives given in Table 13. Tables 14,15,19,23 represents normalized decision matrix data. Tables 11,16,20,24 show the aggregated values. Tables 12,17,21,25 illustrate the score values of alternatives. The ranking of alternatives by using the aggregation operators are presented in Tables 18,22,26.

Table 10

Normalized decision matrix

Symbols	G_1	G_2	G_3
C_1	$(0.5e^{\iota 2\pi(0.55)}, 0.3e^{\iota 2\pi(0.15)})$	$(0.5e^{\iota 2\pi(0.55)}, 0.22e^{\iota 2\pi(0.3)})$	$(0.2e^{\iota 2\pi(0.5)}, 0.1e^{\iota 2\pi(0.5)})$
C_2	$(0.53e^{\iota 2\pi(0.45)}, 0.3e^{\iota 2\pi(0.24)})$	$(0.6e^{\iota 2\pi(0.45)}, 0.23e^{\iota 2\pi(0.2)})$	$(0.4e^{\iota 2\pi(0.57)}, 0.08e^{\iota 2\pi(0.4)})$
C_3	$(0.4e^{\iota 2\pi(0.58)}, 0.2e^{\iota 2\pi(0.21)})$	$(0.3e^{\iota 2\pi(0.51)}, 0.28e^{\iota 2\pi(0.4)})$	$(0.7e^{\iota 2\pi(0.55)}, 0.11e^{\iota 2\pi(0.42)})$
C_4	$(0.54e^{\iota 2\pi(0.52)} 0.33e^{\iota 2\pi(0.47)})$	$(0.7e^{\iota 2\pi(0.52)} 0.24e^{\iota 2\pi(0.4)})$	$(0.2e^{\iota 2\pi(0.58)} 0.09e^{\iota 2\pi(0.4)})$
C_5	$(0.55e^{\iota 2\pi(0.6)}, 0.4e^{\iota 2\pi(0.34)})$	$(0.8e^{\iota 2\pi(0.6)}, 0.2e^{\iota 2\pi(0.21)})$	$(0.5e^{\iota 2\pi(0.59)}, 0.23e^{\iota 2\pi(0.29)})$

Table 11

C*p,q*-$\mathcal{R}$OFN's Aggregated values

Method	Values of alternatives
C_1	$(0.42e^{\iota 2\pi(0.53)}, 0.14e^{\iota 2\pi(0.25)})$
C_2	$(0.51e^{\iota 2\pi(0.50)}, 0.13e^{\iota 2\pi(0.23)})$
C_3	$(0.55e^{\iota 2\pi(0.54)}, 0.14e^{\iota 2\pi(0.28)})$
C_4	$(0.53e^{\iota 2\pi(0.54)}, 0.15e^{\iota 2\pi(0.29)})$
C_5	$(0.64e^{\iota 2\pi(0.59)}, 0.21e^{\iota 2\pi(0.23)})$

Table 12

C*p,q*-$\mathcal{R}$OFN's Score values

Score values
$\dot{\boldsymbol{S}}(C_1) = \mathbf{0.55}$
$\dot{\boldsymbol{S}}(C_2) = \mathbf{0.56}$
$\dot{\boldsymbol{S}}(C_3) = \mathbf{0.57}$
$\dot{\boldsymbol{S}}(C_4) = \mathbf{0.57}$
$\dot{\boldsymbol{S}}(C_5) = \mathbf{0.61}$

Table 13

The C*p,q*-$\mathcal{R}$OFN's ranking values

Ranking values	**Optimal choice**
$C_5 \geq C_4 \geq C_3 \geq C_2 \geq C_1$	$C_5 \rightarrow 1$

Table 14

Normalized decision matrix

Symbols	$\boldsymbol{G_1}$	$\boldsymbol{G_2}$	$\boldsymbol{G_3}$
C_1	(0.85,0.74)	(0.85,0.74)	(0.92,0.86)
C_2	(0.77,0.41)	(0.87,0.71)	(0.92,0.88)
C_3	(0.9,0.75)	(0.83,0.75)	(0.9,0.85)
C_4	(0.84,0.83)	(0.86,0.74)	(0.94,0.87)
C_5	(0.7,0.5)	(0.89,0.72)	(0.79,0.89)

Table 15

Normalized decision matrix

Symbols	$\boldsymbol{G_1}$	$\boldsymbol{G_2}$	$\boldsymbol{G_3}$
C_1	$\left(0.85e^{\iota 2\pi(0.0)}, 0.74e^{\iota 2\pi(0.0)}\right)$	$\left(0.85e^{\iota 2\pi(0.0)}, 0.74e^{\iota 2\pi(0.0)}\right)$	$\left(0.92e^{\iota 2\pi(0.0)}, 0.86e^{\iota 2\pi(0.0)}\right)$
C_2	$\left(0.77e^{\iota 2\pi(0.0)}, 0.41e^{\iota 2\pi(0.00)}\right)$	$\left(0.87e^{\iota 2\pi(0.0)}, 0.71e^{\iota 2\pi(0.0)}\right)$	$\left(0.92e^{\iota 2\pi(0.0)}, 0.88e^{\iota 2\pi(0.0)}\right)$
C_3	$\left(0.9e^{\iota 2\pi(0.0)}, 0.75e^{\iota 2\pi(0.0)}\right)$	$\left(0.83e^{\iota 2\pi(0.0)}, 0.75e^{\iota 2\pi(0.0)}\right)$	$\left(0.9e^{\iota 2\pi(0.0)}, 0.85e^{\iota 2\pi(0.0)}\right)$
C_4	$\left(0.84e^{\iota 2\pi(0.0)}0.83e^{\iota 2\pi(0.0)}\right)$	$\left(0.86e^{\iota 2\pi(0.0)}0.74e^{\iota 2\pi(0.0)}\right)$	$\left(0.94e^{\iota 2\pi(0.0)}0.87e^{\iota 2\pi(0.0)}\right)$
C_5	$\left(0.7e^{\iota 2\pi(0.0)}, 0.5e^{\iota 2\pi(0.0)}\right)$	$\left(0.89e^{\iota 2\pi(0.0)}, 0.72e^{\iota 2\pi(0.0)}\right)$	$\left(0.79e^{\iota 2\pi(0.0)}, 0.89e^{\iota 2\pi(0.0)}\right)$

Table 16

C*p,q*-$\mathcal{R}$OFN's Aggregated values

Method	**Values of alternatives**
C_1	$\left(1e^{\iota 2\pi(0.0)}, 0.68e^{\iota 2\pi(0.0)}\right)$
C_2	$\left(0.87e^{\iota 2\pi(0.0)}, 0.56e^{\iota 2\pi(0.00)}\right)$
C_3	$\left(0.88e^{\iota 2\pi(0.0)}, 0.68e^{\iota 2\pi(0.0)}\right)$
C_4	$\left(0.89e^{\iota 2\pi(0.0)}0.72e^{\iota 2\pi(0.0)}\right)$
C_5	$\left(0.80e^{\iota 2\pi(0.0)}, 0.60e^{\iota 2\pi(0.0)}\right)$

Table 17

C*p,q*-$\mathcal{R}$OFN's Score values

Score values
$\dot{\boldsymbol{S}}(C_1) = \mathbf{0.69}$
$\dot{\boldsymbol{S}}(C_2) = \mathbf{0.64}$
$\dot{\boldsymbol{S}}(C_3) = \mathbf{0.61}$
$\dot{\boldsymbol{S}}(C_4) = \mathbf{0.60}$
$\dot{\boldsymbol{S}}(C_5) = \mathbf{0.59}$

Table 18

The C*p,q*-$\mathcal{R}$OFN's ranking values

Ranking values	**Optimal choice**
$C_1 \geq C_2 \geq C_3 \geq C_4 \geq C_5$	$C_1 \rightarrow 1$

Table 19

Normalized decision matrix

Symbols	G_1	G_2	G_3
C_1	$\left(0.9e^{\iota 2\pi(0.0)}, 0.2e^{\iota 2\pi(0.0)}\right)$	$\left(0.83e^{\iota 2\pi(0.0)}, 0.28e^{\iota 2\pi(0.0)}\right)$	$\left(0.9e^{\iota 2\pi(0.0)}, 0.11e^{\iota 2\pi(0.0)}\right)$
C_2	$\left(0.85e^{\iota 2\pi(0.0)}, 0.3e^{\iota 2\pi(0.00)}\right)$	$\left(0.85e^{\iota 2\pi(0.0)}, 0.22e^{\iota 2\pi(0.0)}\right)$	$\left(0.92e^{\iota 2\pi(0.0)}, 0.1e^{\iota 2\pi(0.0)}\right)$
C_3	$\left(0.84e^{\iota 2\pi(0.0)}, 0.3e^{\iota 2\pi(0.0)}\right)$	$\left(0.86e^{\iota 2\pi(0.0)}, 0.23e^{\iota 2\pi(0.0)}\right)$	$\left(0.94e^{\iota 2\pi(0.0)}, 0.08e^{\iota 2\pi(0.0)}\right)$
C_4	$\left(0.77e^{\iota 2\pi(0.0)} 0.33e^{\iota 2\pi(0.0)}\right)$	$\left(0.87e^{\iota 2\pi(0.0)} 0.24e^{\iota 2\pi(0.0)}\right)$	$\left(0.92e^{\iota 2\pi(0.0)} 0.09e^{\iota 2\pi(0.0)}\right)$
C_5	$\left(0.7e^{\iota 2\pi(0.0)}, 0.4e^{\iota 2\pi(0.0)}\right)$	$\left(0.89e^{\iota 2\pi(0.0)}, 0.2e^{\iota 2\pi(0.0)}\right)$	$\left(0.79e^{\iota 2\pi(0.0)}, 0.23e^{\iota 2\pi(0.0)}\right)$

Table 20

C*p,q*-$\mathcal{R}$OFN's Aggregated values

Method	**Values of alternatives**
C_1	$\left(0.88e^{\iota 2\pi(0.0)}, 0.14e^{\iota 2\pi(0.0)}\right)$
C_2	$\left(0.88e^{\iota 2\pi(0.0)}, 0.14e^{\iota 2\pi(0.0)}\right)$
C_3	$\left(0.89e^{\iota 2\pi(0.0)}, 0.13e^{\iota 2\pi(0.0)}\right)$
C_4	$\left(0.87e^{\iota 2\pi(0.0)} 0.15e^{\iota 2\pi(0.0)}\right)$
C_5	$\left(0.80e^{\iota 2\pi(0.0)}, 0.21e^{\iota 2\pi(0.0)}\right)$

Table 21

score values

Score values
$\dot{\check{S}}(C_1) = \mathbf{0.67}$
$\dot{\check{S}}(C_2) = \mathbf{0.67}$
$\dot{\check{S}}(C_3) = \mathbf{0.67}$
$\dot{\check{S}}(C_4) = \mathbf{0.66}$
$\dot{\check{S}}(C_5) = \mathbf{0.62}$

Table 22

The C*p,q*-$\mathcal{R}$OFN's ranking values

Ranking values	**Optimal choice**
$C_1 \geq C_2 \geq C_3 \geq C_4 \geq C_5$	$C_1 \rightarrow 0.75$

Table 23

Normalized decision matrix

Symbols	G_1	G_2	G_3
C_1	$(0.5e^{\iota 2\pi(0.0)}, 0.3e^{\iota 2\pi(0.00)})$	$(0.5e^{\iota 2\pi(0.0)}, 0.22e^{\iota 2\pi(0.0)})$	$(0.2e^{\iota 2\pi(0.0)}, 0.1e^{\iota 2\pi(0.0)})$
C_2	$(0.53e^{\iota 2\pi(0.0)}, 0.3e^{\iota 2\pi(0.0)})$	$(0.6e^{\iota 2\pi(0.0)}, 0.23e^{\iota 2\pi(0.0)})$	$(0.4e^{\iota 2\pi(0.0)}, 0.08e^{\iota 2\pi(0.0)})$
C_3	$(0.54e^{\iota 2\pi(0.0)} 0.33e^{\iota 2\pi(0.0)})$	$(0.7e^{\iota 2\pi(0.0)} 0.24e^{\iota 2\pi(0.0)})$	$(0.2e^{\iota 2\pi(0.0)} 0.09e^{\iota 2\pi(0.0)})$
C_4	$(0.4e^{\iota 2\pi(0.0)}, 0.2e^{\iota 2\pi(0.0)})$	$(0.3e^{\iota 2\pi(0.0)}, 0.28e^{\iota 2\pi(0.0)})$	$(0.7e^{\iota 2\pi(0.0)}, 0.11e^{\iota 2\pi(0.0)})$
C_5	$(0.55e^{\iota 2\pi(0.0)}, 0.4e^{\iota 2\pi(0.0)})$	$(0.8e^{\iota 2\pi(0.0)}, 0.2e^{\iota 2\pi(0.0)})$	$(0.5e^{\iota 2\pi(0.0)}, 0.23e^{\iota 2\pi(0.0)})$

Table 24

C*p,q*-$\mathcal{R}$OFN's Aggregated values

Method	Values of alternatives
C_1	$(0.88e^{\iota 2\pi(0.0)}, 0.14e^{\iota 2\pi(0.0)})$
C_2	$(0.88e^{\iota 2\pi(0.0)}, 0.14e^{\iota 2\pi(0.0)})$
C_3	$(0.89e^{\iota 2\pi(0.0)}, 0.13e^{\iota 2\pi(0.0)})$
C_4	$(0.87e^{\iota 2\pi(0.0)} 0.15e^{\iota 2\pi(0.0)})$
C_5	$(0.80e^{\iota 2\pi(0.0)}, 0.21e^{\iota 2\pi(0.0)})$

Table 25

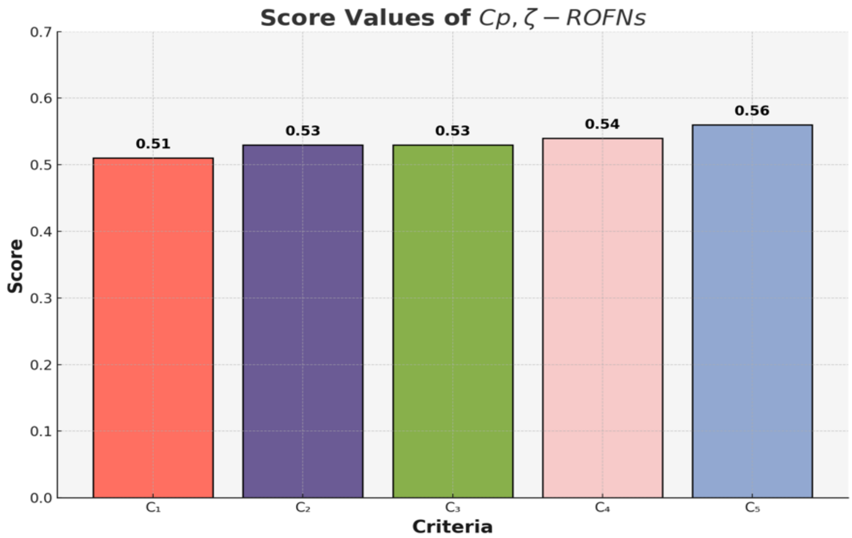
C*p,q*-$\mathcal{R}$OFN's Score values

Score values
$\dot{S}(C_1) = \mathbf{0.51}$
$\dot{S}(C_2) = \mathbf{0.53}$
$\dot{S}(C_3) = \mathbf{0.53}$
$\dot{S}(C_4) = \mathbf{0.54}$
$\dot{S}(C_5) = \mathbf{0.56}$

Table 26

The C*p,q*-$\mathcal{R}$OFN's ranking values

Ranking values	Optimal choice
$C_5 \geq C_4 \geq C_3 \geq C_2 \geq C_1$	$C_5 \to 1$



3.5 Sensitivity Analysis and Discussion

To test the robustness of the proposed model, a sensitivity analysis was conducted by varying the control parameters p, q and γ. These parameters govern the flexibility and strictness of the complex p, q -Rung Orthopair Fuzzy environment. For each variation, the ranking of alternatives was observed and compared to the baseline case. The results demonstrate that while minor variations in these parameters do not significantly alter the top-ranked alternative, certain threshold values do cause changes in ranking positions**,** especially among middle-ranked alternatives. Notably, increasing the value of ppp enhances the influence of higher membership degrees, while higher \q\q\q values provide a more conservative interpretation of membership uncertainty.

Lessons Learned and Practical Guidelines:

1. Lesson 1**:** For problems requiring more inclusive and optimistic evaluations, a lower value of $\boldsymbol{p}$ is preferred, as it allows partial preferences to contribute more significantly to the aggregation.

2. Lesson 2: For highly conservative or risk-averse evaluations, increasing $\boldsymbol{q}$ improves stability by dampening the effect of uncertainty in the membership and non-membership values.

3. Lesson 3: The parameter $\boldsymbol{\gamma}$, when used in the hybrid Hamacher operators, serves as a tuning knob between multiplicative and averaging effects—offering a useful balance between strict aggregation and neutral weighting.

4. Rule of Thumb: If decision-makers are unsure about precise parameter values, selecting $\boldsymbol{p} = \boldsymbol{2}$, $\mathrm{q} = \boldsymbol{1}$, $\boldsymbol{and\ \gamma = 0.5}$ offers a balanced trade-off between optimistic and conservative decision-making.

This analysis confirms the robustness of the proposed model and provides practical insights into parameter tuning, making it a valuable tool for complex decision-making problems involving uncertainty and subjectivity.

4. Conclusion

One significant and effective way to address the conflict between environmental pollution and economic growth is to implement Cleaner Production (CP). Gold miners have used CP to safeguard the environment while extracting resources to support sustainable development. Moreover, the Hamacher Aggregation Operator (HAO) is a traditional type of operator used in the theory of fusion. Its primary characteristic is the ability to model interactions between multiple input arguments.

To explore the properties and applicability of Hamacher-based averaging and geometric operators, we introduced novel Hamacher operators for Complex p, q-Rung Orthopair Fuzzy Numbers (Cp,q-$\mathcal{R}$OFNs). The importance of ordered positions and arguments was also considered in the development of the Hamacher Hybrid Aggregation Operators (HHAOs). We examined the scenarios under which the proposed operators reduce to classical fuzzy, intuitionistic, and Pythagorean environments, demonstrating their generalization capability.

Using the proposed Cp,q-$\mathcal{R}$OFHA operators, we developed a Multi-Attribute Group Decision-Making (MAGDM) algorithm to assess CP alternatives in gold mining. Through a comprehensive numerical study, we verified the feasibility, effectiveness, and robustness of the proposed approach. Sensitivity analysis further revealed the influence of parameters $p, q$and γ on the final rankings. Comparative results indicated that the proposed model outperforms existing approaches in terms of precision, flexibility, and robustness.

Limitations and Future Work:

While the proposed model performs well in the context of gold mining, it has certain limitations. The current framework is restricted to Cp,q-$\mathcal{R}$OFNs and assumes known weights and input values. In future research, the model can be extended to more generalized fuzzy environments such as Pythagorean complex fuzzy sets or Neutrosophic complex fuzzy sets**.** Furthermore, the proposed aggregation operators and MAGDM methodology can be applied to diverse domains such as healthcare decision systems**,** sustainable supply chain management**,** and smart city planning to test its adaptability and scalability across sectors.

Author Contributions
All authors equally contributed. “Conceptualization, T. and A.N.; methodology, T.; software, T.; validation, T. and A.N.; formal analysis, T. and A.N.; investigation, T. and A.N.; resources, T. and A.N.; data curation, T. and A.N.; writing—original draft preparation, A.N.; writing—review and editing, T. and A.N.; visualization, T. and A.N.; supervision, T. and A.N.; project administration, T. and A.N.

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No Data Availability.

Conflicts of Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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